1. Trigonometry - Textbooks, I.

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ELEMENTS

OF

ANALYTIC TRIGONOMETRY,

PLANE AND SPHERICAL.

BY

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REMARK:—As it is essential, in an elementary book, that the expressions be appropriate to the subject, and even to the local usages of the language of the science, of whose elements it treats, as far as the latter can be admitted without diminishing the precision of the expressions; and as this would require the author to be a native of the country in whose language the treatise is published, which is not my case; my friend Professor Renwick, so advantageously known to the public by his own works, has done me the favour to translate into English the manuscript of this work, which I drew up in French. We considered this as the surest means of obtaining the desired object of bringing this work before the public in a style unembarrassed by other idioms, and whose expressions would be adapted, not only to the language itself, but to established usages of this science.

INTRODUCTION.

Mathematical science must, from its very nature, have taken its rise in the simple inspection of geometric figures. The abstractions, upon which the calculus is founded, and whose great extension and generalization has produced the analytic method, must have arisen at a later period, as the product of a higher cultivation of the powers of the mind.

During the period that geometry constituted the principal part of mathematical science, trigonometry was necessarily treated of by the synthetic methods applicable to that branch of the science; and the solution of its several problems, attained by mere construction. Calculation was subsequently introduced, when the means were discovered, by which numbers could be applied to express the relations of quantities, which appear so different in their respective natures, as linear dimensions and angles.

Analysis, so bold in its steps and so universal in its methods, which has carried mathematical science to results the most general, and of such extensive and useful consequences, has naturally changed the mode of proceeding in trigonometry, as well as in other departments of mathematics. It is therefore necessary now, in order to study trigonometry in a truly scientific way, to treat of it in the most general manner; and, proceeding from principles the most general, yet at the same time the most simple and elementary, to found upon them a complete system; whose results may be fitted for universal application.

It is not necessary to enter into all the details, that are

the necessary consequences of such a system, in endeavouring to attain this object; they will not escape the researches of him, who has made himself master of the system itself.

With such views the present elementary treatise has been drawn up; and it is not necessary to explain the difference, that exists between it and the various other manners in which trigonometry has been treated of.* The principles upon which it is grounded are the following.

As straight lines and angles, or portions of the circumference of a circle, are incommensurable quantities, they cannot be directly compared. But the ratio between two of the sides of a right angled triangle, will determine the magnitude of the acute angles; the third angle being always given, in consequence of the primitive condition of rectangularity in the triangle. This ratio then is the true and only means by which angles may be compared with straight lines.

The names that are given to the several ratios, that exist among the sides of a right angled triangle, taken by pairs, are purely conventional, although the terms have in part been deduced from geometric considerations, having reference to the circle. But it is of the greatest importance carefully to avoid confounding the lines, that correspond to these ratios, or trigonometrical functions, when represented in a circle, with these ratios themselves.†

* It was the desire of introducing into the course of mathematics at the United States' military academy at West-point, the most useful mode of instruction in this branch, that led me to the preparation of this work, as early as the year 1807.

† The term *sine* owes its origin simply to a contraction in writing *semissis cordæ;* when, in the middle age, instead of the chords of angles, that were formerly employed in calculation, their halves were introduced, writing merely *sm;* and *co-sm* for *complementi semissis cordæ,* the tangent is represented geometrically by the line touching the circle without cutting it, and is the only appropriate denomination taken from the circle. The prolongation of the radius, until it cut the tangent, has been called *secant,* which is a perversion of the name given in geometry to a line that cuts the circle without passing through the centre. The addition, *co,* before each of these names, refers them, as in the case of the sine, to the complementary angle.

Setting out then from the primary definitions of the ratios, that exist between the three lines that form a right angled triangle, combining them, simply, and by their squares, according to the properties of right angled triangles, deduced from the most elementary geometry; (Euclid I. p. 47) we shall obtain, by means of the four fundamental rules of ordinary arithmetic, applied algebraically to these elementary expressions, a series of elementary formulæ.

These formulæ give the solution of every possible case of right lined rectangular trigonometry; and furnish a general table for the reduction of the several trigonometric functions to each other; similar in its nature and application to the common multiplication table. In this way we are furnished with a system of quantities, whose relative relations are determined; the fruitful source of every possible combination.

By the simple consideration of two angles united by juxta or super-position, (a method employed in elementary geometry,) applying the same elementary process, founded upon the principles previously employed, the second step in the system is made. This step furnishes the general principles of the combination of the trigonometric functions of the sum and difference of two or more angles.

The same system of combination used before, applied to this second series of formulæ; with different assumptions in relation to the relative value of the two angles; and also when they are supposed to have a constant determinate value; leads to all the various formulæ that can be desired; which are given in regular tables systematically arranged; and which may be referred to with the greatest readiness.

This mode of proceeding appears to lead to the desired aim with the least labour of intellect, and thus in the most easy way to the final end; which is, to present to the reader a full system of this branch of mathematics, in such a way, as to furnish every necessary element for the solutions of trigonometry, both plane and spherical; and for the use of

analysis in general, in its numerous applications to geometry, and to transcendant quantities.

It is with a similar view that the chapter which points out the mode of making use of the trigonometric functions in the integral calculus, and chiefly for the purpose of transforming the formulæ to fit them for integration, has been inserted. Trigonometric differentials are however omitted; they would require the application of the differential calculus, the knowledge of which is not to be presumed in the student of elementary trigonometry. It was thought more expedient to defer this part to a subsequent extension of the course of trigonometry; that should at the same time present its applications, and several other problems; both theoretic and practical, (and which will form the sequel of this elementary treatise, if it be approved by the public.)

It is thought: that the method of applying the trigonometric functions to algebra, by a change of the formulæ, such as to admit the use of logarithms, to change addition or subtraction into multiplication, &c. a method as simple as useful, is sufficiently explained by the use which is made of it in the course of this treatise. For this reason it has not been separately considered, as it might have been, in applying it to the solution of equations of the second and third order, &c. But when the applications, that are actually made of it in this treatise, are well understood, those to other cases will be also intelligible.

Although, for the reasons already stated, the explanation of the ingenious methods, that may be employed in the construction of trigonometric tables, both uatural and logarithmic, is not admitted into this plan; it has been thought proper to explain their fundamental principles; in order to complete the system.

The considerations that have reference to the radius of the circle, are not given, except where it becomes necessary to employ them; thus the student does not find himself embar-

rassed with them in those parts, where they could answer no other purpose, but that of confusing his ideas.

These principles being established, the solution of all the cases of oblique angled plane triangles follows, as their most obvious application; and the use that is made of the forms that are given to the trigonometric functions, in reducing the calculations to logarithms, is a sufficient introduction to this method.

When the analytical method is applied to spherical trigonometry, it is obviously proper, first to expose some of the immediate consequences of the theorems of solid geometry, in their application to the sphere, and then to express them in the form of trigonometric functions. Setting out in this manner immediately from solid geometry, we avoid, as will be seen, all the delay and difficulty, which would attend the introduction of spherics in the abstract.

The combinations of the parts of the right angled triangles, that constitute the elements of a spherical triangle, considered by the method of trigonometric functions, also forms in this part of the work the principle whence all the elementary formulæ of spherical trigonometry are deduced. The combinations of these give all the solutions, that this branch of trigonometry demands.

It has been thought that the continuation of the method previously used, was also in this part of trigonometry preferable to the introduction of another, although equally good in itself; for it is with methods in mathematics, as with style in ordinary writings: that author is most easily understood, who expresses himself in one uniform and fixed manner; while a change in the method of expression naturally introduces uncertainty in the apprehension of the sense of the writer.

For a similar consideration, the means of deduction, or the representation of the different subjects, have not been multiplied: an elementary book need not give all that the author

knows on the subject, but only all that is necessary to constitute a complete system.

The mechanical arrangement of a calculation may conduce to its accuracy, and to the ease of revising it, in case of need. It is with this, as with order in all matters of business; it is proper in the beginning to acquire good habits, which practice will render easy. The numerous and frequently complicated operations of trigonometry have especially need of such a precaution.

As an introduction to this practical part, there will be introduced at the close of this treatise an example of the calculation of each formula in an order the most concise, and most applicable to practice. In the complicated calculations of the practical application of trigonometry, it is useful to have forms of the process in blank, containing the order and denominations of the operations, and having a blank space sufficient for the insertion of the numbers. In this way the calculations may be reduced to an operation purely mechanical, in which no one of the necessary elements can possibly be omitted. This method has been long used in great geodetic works, and in navigation.

This treatise is then naturally divided into four parts.

1st Part. Analysis of the Trigonometric Functions.

2d " Oblique angled Plane Trigonometry.

3d " Spherical Trigonometry.

4th " Examples of Calculation of the Formulæ of Plane and Spherical Trigonometry.

It only remains to give a few details in relation to some elementary principles made use of; and to such as are purely conventional, that it will become necessary to employ in this treatise.

Elementary Geometry teaches us that all the angles around any one point are together equal to four right angles; it follows that the circumference of a circle contains also four right angles.

The ordinary mode of expressing a right angle, is, $= \llcorner R$.

The division of the circumference of a circle is, from its very nature, conventional. Three different divisions have been used, at different times, and with different views; the most ancient of these enjoys the right derived from its priority of occupation. This is the division of the circumference into 360 equal parts, or *degrees*: each of these is divided into 60 equal parts, called *minutes*, (minutæ partes;) and these again into 60 parts, called *seconds*, (partes minutæ secundæ.) In the same manner we might proceed to obtain *thirds*, *fourths*, &c.; but instead of this, it is the custom at the present day, to represent the magnitudes of parts less than seconds, in the decimals of that denomination.

This division may therefore be represented in an algebraic form, (marking degrees by a small cypher above the numbers, minutes, by a single line, seconds, by two lines, &c.) as follows, viz:

$$\pi = 360^\circ = 4 \llcorner R;\ 90^\circ = \llcorner R;\ 1^\circ = 60';\ 1' = 60''.$$

This furnishes the principle of the method of reduction, or transformation, of one denomination into another; and we might express the whole of the circumference in the following manner, viz:

$$\pi = 3 \llcorner R + 89^\circ\ 59'\ 60''.$$

The division of the fourth part of the circle or quadrant into 100°, with decimal subdivisions, has been several times attempted; in consequence of the usefulness such a division would possess, in all geodetic operations, when combined with the corresponding decimal metrical system.

The division of the quadrant into 96^c has been employed by some of the best artists, in the graduation of great astronomical instruments. It is very advantageous in this process, because all the subdivisions, down to the single degree, may be obtained by the continual bisection of an arc, whose cord is equal to the radius of the circle; or in this division 64°, making in the ordinary division 60°.

The common division into 360° will be used in this treatise.

In order to show that the difference between two quantities is to be taken, in such a way that the result shall be always a positive quantity, which ever of the two be the greater, we shall use the sign $\backsim$, or an S lying horizontally.

The complement of an angle is that angle, which, when added to it, makes their sum a right angle. Thus the angle, b, has for its complement $90°-b, = \llcorner R - b$.

The supplement of an angle is that angle, which, added to it, makes the sum equal to $2 \llcorner R = 180°$. Thus the angle, b, has for its supplement, $2 \llcorner R - b, = 180° - b$.

All other methods of notation, and the signs made use of, are derived from Algebra.

PART I.

ANALYSIS OF TRIGONOMETRIC FUNCTIONS.

CHAPTER I.

Deduction of the First elementary Formulæ.

§ 1. ANALYTIC TRIGONOMETRY is one of the problems of Algebra applied to Geometry; it not only comprises all those solutions that are necessary to find the unknown parts of triangles from those which are known; but furnishes a series of formulæ and analytical expressions, that may be finally applied to Analysis in general; and which constitute a peculiar species of quantities called Trigonometric Functions. Considered in this point of view, it forms one of the most important branches of analytic mathematics.

§ 2. If the three angular points of a triangle be considered as lying in the same plane, in which, therefore, the lines which join these points are likewise situated, the triangle becomes the subject of the investigation of Plane Trigonometry. Elementary geometry makes us acquainted with the principles of equality and proportion that exist between them under certain relations of their several parts, and trigonometry employs these principles as the basis of its researches.

§ 3. If the angular points of the triangle be considered as not in the same plane, the triangle becomes, generally speaking, the subject of the investigation of Spherical Trigonometry, as it is referred to the curved surface generated by the revolution of the circumference of a circle around its diameter, or the surface of a sphere; its properties are derived from solid geometry; and it is the only curved sur-

face that is considered in the elementary part of that branch of mathematics.

§ 4. It is evident that, in the extension of the subject, there may be a separate species of trigonometry for every possible variety of surface generated by the revolution of a re-entering curve. The equation of the radius of the curve would be an essential element of the resulting trigonometry; as, for instance, an ellipsoidic or spheroidic trigonometry. But this case requires a more complicated analysis; it is more detailed in its investigations, and consequently less general in its applications: it therefore cannot belong to elementary mathematics.

§ 5. The elementary trigonometric functions are the ratios that exist between the three sides forming a right angled plane triangle; or, in other words, the quotients that arise from dividing any one of them by either of the two others. There are not, therefore, necessarily more than three such functions, to which are added their inverse ratios. These several functions are known by names, whose origin and signification are of no importance; but it is the more important, that we fully and precisely understand their value, and mutual relations.

The combination of these ratios gives the whole of that multitude of trigonometric functions, that enable us to solve every question in trigonometry, and which are perpetually applied in analysis.

§ 6. Let, ABC, (figure 1) be a plane triangle, right angled at A; the sum, therefore, of the two other angles, $B+C=\llcorner R=90°$. They are, consequently, each the difference between the other and a right angle. This relation of these two angles being the complement, as has been previously stated, we have, according to the division of the circle into 360°, $B=90°-C$; and $C=90°-B$.

The theorem of elementary geometry known by the name of Pythagoras (Euclid, Book I. prop. 47) gives the following relations:

$$BC^2 = AB^2 + AC^2, \text{ whence}$$
$$AB^2 = BC^2 - AC^2, \text{ and}$$
$$AC^2 = BC^2 - AB^2.$$

To simplify these expressions, let BC=h, AB=k, AC=d, and we have

$$h^2 = k^2 + d^2$$
$$k^2 = h^2 - d^2$$
$$d^2 = h^2 - k^2$$

which determine the relations between the sides of a right angled plane triangle, in terms of their squares.

§ 7. To these properties of a right angled triangle, given in elementary geometry, trigonometry adds the expressions that denote the *ratios* of the several sides; or rather, it gives to each of these *ratios* a specific name, as follows, viz:

The ratio,	or the quotient,	is called	A
$AC : BC$,	$\frac{d}{h}$	$=$sine B=cosine $(90^\circ - B)$=cosine C	1
$AB : BC$,	$\frac{k}{h}$	$=$cosine B=sine $(90^\circ - B)$=sine C	2
$CA : BA$,	$\frac{d}{k}$	$=$tangent B=cotangent$(90^\circ - B)$=cotangent C	3
$BA : CA$,	$\frac{k}{d}$	$=$cotangent B=tangent $(90^\circ - B)$=tangent C	4
$BC : AB$,	$\frac{h}{k}$	$=$secant B=cosecant $(90^\circ - B)$=cosecant C	5
$BC : AC$,	$\frac{h}{d}$	$=$cosecant B=secant $(90^\circ - B)$=secant C	6

It is evident from inspection, that the prefix, co, before the names sine, tangent, secant, show that the relations of the quantities are the same when they are referred to the complementary angle, as when with their simple names, they are considered in relation to the angle itself.

It is also evident that the three last ratios are the inverse of the three first. They are consequently much less used than the three first, particularly the two last: these are, in-

deed, at present entirely neglected, as well as the terms, versed sine, and co-versed sine; having all become useless, in consequence of the great simplification that has taken place in trigonometric formulæ.

§ 8. Combining the primitive formulæ thus found, or determined, by their multiplication and division, and comparing the results with the simple formulæ, or definitions, to which the products or quotients are equal, we obtain a series of functions, or formulæ, that constitute what may be called the multiplication table of analytic trigonometry. Thus:

B By the multiplication of

1 A No 1 into No 6 or $\frac{d}{h}, \frac{h}{d} = \text{sine } B \text{ cosec } B = 1$

2 2 5 $\frac{k}{h}, \frac{h}{k} = \cos B \sec B = 1$

3 3 4 $\frac{d}{k}, \frac{k}{d} = \tan B. \cot B = 1$

4 1 4 $\frac{d}{h}, \frac{k}{d} = \frac{k}{h} = \text{sine } B \cot B = \cos B$

5 2 3 $\frac{k}{h}, \frac{d}{k} = \frac{d}{h} = \text{cosine } B \tan B = \text{sine } B$

6 1 5 $\frac{d}{h}, \frac{h}{k} = \frac{d}{k} = \text{sine } B \sec B = \tan B$

7 2 6 $\frac{k}{h}, \frac{h}{d} = \frac{k}{d} = \cos B \text{ cosec } B = \cot B$

By the division of

8 H 1 by No 2 or $\frac{d}{h} : \frac{k}{h} = \frac{d}{k} = \frac{\text{sine } B}{\cos B} = \tan B$

9 2 1 $\frac{k}{h} : \frac{d}{h} = \frac{k}{d} = \frac{\cos B}{\text{sine } B} = \cot B$

10 1 3 $\frac{d}{h} : \frac{d}{k} = \frac{k}{h} = \frac{\text{sine } B}{\tan B} = \cos B$

No 3 by No 1 or		$\frac{d}{k} : \frac{d}{h} = \frac{h}{k} = \frac{\tan B}{\text{sine } B} = \sec B$	11
2	4	$\frac{k}{h} : \frac{k}{d} = \frac{d}{h} = \frac{\cos B}{\cot B} = \text{sine } B$	12
4	2	$\frac{k}{d} : \frac{k}{h} = \frac{h}{d} = \frac{\cot B}{\cos B} = \text{cosec } B$	13
3	5	$\frac{d}{k} : \frac{h}{k} = \frac{d}{h} = \frac{\tan B}{\sec B} = \text{sine } B$	14
5	3	$\frac{h}{k} : \frac{d}{k} = \frac{h}{d} = \frac{\sec B}{\tan B} = \text{cosec } B$	15
4	6	$\frac{k}{d} : \frac{h}{d} = \frac{k}{h} = \frac{\cot B}{\text{cosec } B} = \cos B$	16
6	4	$\frac{h}{d} : \frac{k}{d} = \frac{h}{k} = \frac{\text{cosec } B}{\cot B} = \sec B$	17
5	6	$\frac{h}{k} : \frac{h}{d} = \frac{d}{k} = \frac{\sec B}{\text{cosec } B} = \tan B$	18
6	5	$\frac{h}{d} : \frac{h}{k} = \frac{k}{d} = \frac{\text{cosec} B}{\sec B} = \cot B$	19

If the combinations producing squares were admitted into this table, it would become more extensive, but it is not considered proper to introduce them here, as they may be considered with more propriety as consequences.

It will also be observed that some of the above results are already repetitions, for they may be considered as algebraically contained in preceding ones; but this, being exactly analogous to what occurs in the common multiplication table, has been admitted, in the same way as, in a complete multiplication table, two equal products, say, for instance, 3 times 3, and 4 times 3, are introduced, to accustom the beginner to their equality.

§ 9. If we apply the three expressions deduced from the 47th Prop. of Euclid, Book I. given in § 6, viz:

$$h^2 = d^2 + k^2;\ d^2 = h^2 - k^2;\ k^2 = h^2 - d^2;$$

to those found in the series, A, making use of the expressions that have the same denominator, and reducing the numerators, resulting from the addition or subtraction of their squares, we obtain a new series of formulæ, that give the relations of the squares of the several functions; viz:

C The sum of the squares of

1 1 and 2, or $\dfrac{d^2}{h^2} + \dfrac{k^2}{h^2} = \dfrac{d^2+k^2}{h^2} = \dfrac{h^2}{h^2} = 1 = \sin{}^2 B + \cos{}^2 B$

The difference of the squares of

2 5 and 3, or $\dfrac{h^2}{k^2} - \dfrac{d^2}{k^2} = \dfrac{h^2-d^2}{k^2} = \dfrac{k^2}{k^2} = 1 = \sec{}^2 B - \tan{}^2 B$

3 6 and 4, $\dfrac{h^2}{d^2} - \dfrac{k^2}{d^2} = \dfrac{h^2-k^2}{d^2} = \dfrac{d^2}{d^2} = 1 = \operatorname{cosec}{}^2 B - \cot{}^2 B$

From these equations are obtained, by simply transposing the terms, the following, which are of very frequent use in trigonometric calculations.

4 $\sin{}^2 B = 1 - \cos{}^2 B$

5 $\cos{}^2 B = 1 - \sin{}^2 B$

6 $\sec{}^2 B = 1 + \tan{}^2 B$

7 $\operatorname{cosec}{}^2 B = 1 + \cot{}^2 B$

8 $\tan{}^2 B = \sec{}^2 B - 1$

9 $\cot{}^2 B = \operatorname{cosec}{}^2 B - 1$

By equalizing the first three results, it is also evident that

10 $\sin{}^2 B + \cos{}^2 B = \sec{}^2 B - \tan{}^2 B = \operatorname{cosec}{}^2 B - \cot{}^2 B = 1$

and

11 $\sec{}^2 B - \operatorname{cosec}{}^2 B = \tan{}^2 B - \cot{}^2 B$

with their several consequences.

§ 10. Combining the two series of formulæ, B, and, C, by simply substituting the roots taken from, C, in the formulæ

of **B**, we obtain another series of formulæ, of frequent use in the application of logarithms to trigonometrical calculations, and in the integral calculus.

From what has been observed, and has been already shown, it is sufficient to give these for sines, cosines, and tangents; for which the following values will be successively obtained.

By B	and the substitution from		D
No. 5	C No. 6	$\text{sine } B = \cos B\,(\sec^2 B-1)^{\frac{1}{2}}$	1
1	7	$= \dfrac{1}{(1+\cot^2 B)^{\frac{1}{2}}}$	2
14	6	$= \dfrac{\tan B}{(1+\tan^2 B)^{\frac{1}{2}}}$	3
12	7	$= \dfrac{\cos B}{(\text{cosec}^2 B-1)^{\frac{1}{2}}}$	4
5	5 and 6	$= (1-\sin^2 B)^{\frac{1}{2}}\,(\sec^2 B-1)^{\frac{1}{2}}$	5
14	2 and 6	$= \dfrac{(\sec\ B-1)^{\frac{1}{2}}}{(1+\tan^2 B)^{\frac{1}{2}}}$	6
12	5 and 6	$= \dfrac{(1-\sin^2 B)^{\frac{1}{2}}}{(\text{cosec}^2 B-1)^{\frac{1}{2}}}$	7
By B	substituting from		
No. 4	C 9	$\text{cosine } B = \text{sine } B\,(\text{cosec}^2 B-1)^{\frac{1}{2}}$	8
2	6	$= \dfrac{1}{(1+\tan^2 B)^{\frac{1}{2}}}$	9
10	6	$= \dfrac{\text{sine } B}{(\sec^2 B-1)^{\frac{1}{2}}}$	10
16	9	$= \dfrac{(\text{cosec}^2 B-1)^{\frac{1}{2}}}{\text{cosec } B}$	11
4	4 and 9	$= (1-\sin^2 B)^{\frac{1}{2}}\,(\text{cosec}^2 B-1)^{\frac{1}{2}}$	12

	By B	substituting from	
13	10	C 4 and 6	$= \dfrac{(1-\cos^{2} B)^{\frac{1}{2}}}{(\sec^{2} B-1)^{\frac{1}{2}}}$
14	16	7 and 9	$= \dfrac{(\operatorname{cosec}^{2} B-1)^{\frac{1}{2}}}{(1+\cot^{2} B)^{\frac{1}{2}}} = \dfrac{\cot B}{(1+\cot^{2} B)^{\frac{1}{2}}}$
15	No. 3	No. 9	$\tan B = \dfrac{1}{(\operatorname{cosec}^{2} B-1)^{\frac{1}{2}}}$
16	6	8	$= \sin B\,(1+\tan^{2} B)^{\frac{1}{2}}$
17	8	5	$= \dfrac{\sin B}{(1-\sin^{2} B)^{\frac{1}{2}}}$
18	18	6	$= \dfrac{(1+\tan^{2} B)^{\frac{1}{2}}}{\operatorname{cosec}^{2} B}$
19	6	4 and 6	$= (1-\cos^{2} B)^{\frac{1}{2}}\,(1+\tan^{2} B)^{\frac{1}{2}}$
20	8	4 and 5	$= \dfrac{(1-\cos^{2} B)^{\frac{1}{2}}}{(1-\sin^{2} B)^{\frac{1}{2}}}$
21	18	6 and 7	$= \dfrac{(1+\tan^{2} B)^{\frac{1}{2}}}{(1+\cot^{2} B)^{\frac{1}{2}}}$

It is evident, that the formulæ for the sine will give those for the cosecant, by merely changing the denominators into numerators, and the numerators into denominators; or, in other words, by expressing the inverse ratio of the sine. In like manner, by performing a similar operation, the values of the cosine will give those for the secant, and those of the tangent the values of the cotangent.

CHAPTER II.

Solution of right angled plane Triangles; Values and Algebraic signs of certain Trigonometric Functions.

§ 11. The formulæ of the preceding chapter are evidently true whatever be the magnitude of the angle *B*, and the ratio for a given angle being given by any one of the functions of the series A, the determination of the value of any one of the lines, *h*, *d*, *k*, will, it is manifest, give the value of the two others.

From this it results, that these formulæ contain the solution of every possible case of a right angled plane triangle. It will suffice for this purpose to make choice of that trigonometric function, in the equation of which, the known quantity is in the denominator of the fraction expressing it, and the unknown quantity in the numerator; and to multiply the trigonometric function of the corresponding angle by the denominator of the fraction; to obtain for result the unknown quantity which is represented by the numerator. For, every ratio being a fraction, or quotient, representing the relative value of two quantities, in which the denominator points out the value of each of the parts; the multiplication of the quotient by the absolute value of all the parts, must present in the result the absolute value of the numerator. This principle is evident from the manner in which the trigonometric functions have been deduced, and is general; it would therefore be useless to enter into any detail.

§ 12. In correspondence with the general principle just stated, the numerical values of these several quotients have been calculated, for all angles from, 0°, to 90°, on the supposition that the value of the denominator is constantly unity; they are therefore directly applicable by means of the rule just given.

As in a right angled triangle one of the acute angles is always the complement of the other, it follows: that when either of them is half a right angle or $= 45°$, the lines, k, and, d, becoming equal, their trigonometric functions of corresponding denomination are also equal, that is to say:

$$\text{sine} = \text{cosine}$$
$$\text{tangent} = \text{cotangent}$$
$$\text{secant} = \text{cosecant}$$

And as, on the angle becoming greater than 45°, the complementary angle takes, in succession, every value of the primitive angle, in an inverted order, it follows: that in an angle between 45° and 90°, the simple change of any one of the above denominations of functions into its corresponding one will give the function sought. For this reason it is only necessary to calculate the value of the sines, cosines, tangents, and cotangents, from 0° to 45°, in order to obtain every other value that is necessary.

§ 13 Let it now be supposed, that any line, $BC=h$, (figure 2) take successively all possible positions around the point, B, so as to form in relation to a fixed line, BA, successively, all the angles from, 0°, to, 360°, in which last position it will again coincide with, 0°, and if we conceive a perpendicular to fall in any position of the line from a point, C, taken at any distance whatsoever from the point, B, upon the line, BA, produced indefinitely on either side of the point, B; and if, according to the constant supposition in geometry, we assign to this line, and to the perpendicular, the proper algebraic signs, to show their direction in relation to the point, B, giving the sign, $+$, to those positions of the lines, d, and k, that correspond in their direction with their primitive position, and the sign, $-$, where they are in an opposite direction; there will result all the variations of value, in quantity and in sign, that these elementary functions can possibly assume.

In order to show this more clearly: Let, BC, BC', BC'', BC''', (figure 2) be several successive positions of this line, in the four right angles, which are contained around the

point, B, the lines, d, and, k, will take the signs assigned to them in the figure, and the signs of the fundamental trigonometric functions contained in the series A, will always be determined, upon the general and simple principle, that serves to determine the signs in algebra; that is to say, that like signs produce, $+$, and unlike ones, $-$. If therefore we compare with the formulæ, the lines, k, and, d, of the figure, in regard to their respective positions, it will be found: that, supposing all the functions within the first right angle to be positive, we shall have in the

2d right angle the,	sines, and, cosecants, $+$,	the other functions, $-$,
3d	tangent, and, cotangent, $+$,	, $-$,
4th	cosine, and, secant, $+$,	, $-$,

§ 14 In the passage of, h, from one quadrant to another, as well as in its first position, the lines, d, and, k, become alternately equal to, 0, and to, h, itself. In these cases they evidently acquire their least and greatest possible values.

If, therefore, we suppose, $h=1$, and use, π, to represent the entire circumference of a circle, the point, 0°, or the origin of the angles, will be represented by, 0π, the first quadrant or right angle, by, $\frac{1}{4}\pi$, and so on. Hence the values of the trigonometric functions in these four principal positions, when expressed in terms of, π, will assume the following values, viz: E

For, 0π, we shall have, $d=0$; and, $k=1$ 1

which gives $\frac{d}{h}=\frac{0}{1}=\sin 0\pi=0$

$$\frac{k}{h}=\frac{1}{1}=\cos 0\pi=1$$

$$\frac{d}{k}=\frac{0}{1}=\tan 0\pi=0$$

$$\frac{k}{d}=\frac{1}{0}=\cot 0\pi=\text{infinit}$$

$$\frac{h}{d} = \frac{1}{0} = \text{cosec } 0\pi = \text{infinit}$$

$$\frac{h}{k} = \frac{1}{1} = \sec 0\pi = 1$$

2 For, $\frac{1}{4}\pi$, we have, $d = 1$, and, $k = 0$,

giving $$\frac{d}{h} = \frac{1}{1} = \sin \tfrac{1}{4}\pi = 1$$

$$\frac{k}{h} = \frac{0}{1} = \cos \tfrac{1}{4}\pi = 0$$

$$\frac{d}{k} = \frac{1}{0} = \tan \tfrac{1}{4}\pi = \text{infinit}$$

$$\frac{k}{d} = \frac{0}{1} = \cot \tfrac{1}{4}\pi = 0$$

$$\frac{h}{d} = \frac{1}{1} = \text{cosec } \tfrac{1}{4}\pi = 1$$

$$\frac{h}{k} = \frac{1}{0} = \sec \tfrac{1}{4}\pi = \text{infinit}$$

3 For, $\frac{2}{4}\pi$, we have, $d = 0$; and, $(k = -1,)$

giving $$\frac{d}{h} = \frac{0}{1} = \text{sine } \tfrac{2}{4}\pi = 0$$

$$\frac{-k}{h} = \frac{-1}{1} = \cos \tfrac{2}{4}\pi = -1$$

$$\frac{d}{-k} = \frac{0}{-1} = \tan \tfrac{2}{4}\pi = -0$$

$$\frac{-k}{d} = \frac{-1}{0} = \cot \tfrac{2}{4}\pi = -\text{infinit}$$

$$\frac{h}{d} = \frac{1}{0} = \text{cosec } \tfrac{2}{4}\pi = +\text{infinit}$$

$$\frac{h}{-k} = \frac{1}{-1} = \sec \tfrac{2}{4}\pi = -1$$

For $\frac{3}{4}\pi$ we have, $d = -1$, and, $k = 0$, 4

$$\text{giving} \quad \frac{-d}{h} = \frac{-1}{1} = \sin \tfrac{3}{4}\pi = -1$$

$$\frac{k}{h} = \frac{0}{1} = \cos \tfrac{3}{4}\pi = 0$$

$$\frac{-d}{k} = \frac{-1}{0} = \tan \tfrac{3}{4}\pi = -\text{ infinit}$$

$$\frac{k}{-d} = \frac{0}{-1} = \cot \tfrac{3}{4}\pi = -0$$

$$\frac{h}{-d} = \frac{1}{-1} = \operatorname{cosec} \tfrac{3}{4}\pi = -1$$

$$\frac{h}{k} = \frac{1}{0} = \sec \tfrac{3}{4}\pi = \text{infinit} \qquad (*)$$

§ 15. It is evident, from what has been said in the two sections immediately preceding, that all the elementary trigonometric functions may be represented in a circle, whose radius is, $h = 1$; and that they will always form proper or improper fractions of this unit, from 0, to infinity.

In fig. 3, let, B, be the centre of the circle, whose radius, $h = 1$, BA, and, Ba, two radii at right angles to each other, that contain the first quadrant; the points, C, C', &c. the successive intersections of, h, with the circumference in the four quadrants; the perpendiculars let fall from the points C, C',

(*) The expression $\frac{1}{0}$ which is here seen to result from the division of the different lines gives the best idea of what is called infinity; for it appears as a ratio (or relative quantity) such as it would exceed the power of any number to express. The sign commonly used for it in analysis is, ∞, or an 8 placed horizontally.

&c. upon the radius BA, produced upon the other side of B, will represent, both in magnitude and algebraic sign, in relation to, $h = 1$, the sines of the angles ABC, ABC', &c. while the parts of the line BA, intercepted between the perpendiculars and the point, B, will represent the several cosines of the same angles.

Draw from C, a line parallel to AB, until it intersect the line Ba, the lines, Cl, and Bl, are equal to Bg, and Cg, each to each; whence it is manifest, that, as the angle aBC, is the complement of ABC, we have

$$\text{sine } ABC = \cos aBC$$
$$\cos ABC = \sin aBC$$

In the same manner, if we draw from the points A, and a, perpendiculars, upon BA, and Ba, produced in either direction from A, and a, the line BC, BC', &c. produced on either side of B, will cut these perpendiculars in points, such as c, c', e, e', and Ac, will represent the tangent; ac' the cotangent, Bc the secant, Bc' the cosecant of the angle ABC; and in these functions of the angle, ABC, the same relation takes place with respect to the exchange of the denominations of these functions, that we have seen to occur in regard to the sines and cosines, &c. of this angle and its complement aBC, for we have

$$\tan ABC = \cot aBC$$
$$\cot ABC = \tan aBC$$
$$\sec ABC = \operatorname{cosec} aBC$$
$$\operatorname{cosec} ABC = \sec aBC$$

The figure shows in what manner the signs of these quantities are affected in the four quadrants; attention being paid to the principle, that A, and a, are always the points from which the tangents are considered to be drawn in either direction, in which they can cut the produced radius, or h. We must be careful here to avoid falling into the error of supposing a change of sign in h; the radius of a circle can never be any thing but a positive quantity; it is only the effect of its position upon the perpendiculars, considered in relation to

the directions of BA, and Ba, which depend for their sign upon the position of h, in the several quadrants, that can be affected by different signs; for in nature, and consequently in mathematics, every efficient cause is positive, while it is only its effect, in regard to a required result, that may become negative.

CHAPTER III.

Fundamental Trigonometric Functions of the Sum, and Difference of two Angles.

§ 16. ***Problem.*** To find the sine and côsine of the sum and difference of two angles, their respective sines and cosines being given.

Let DBC, and, ABC, (in figures 4, and 5,) be the two angles, placed upon the common line, BC, in such a manner that the angle, ABD, may represent their sum, (in figure 4,) or difference, (in figure 5,) when, ABD, represents the sum, the two angles will then each fall without the other; when it represents their difference, the less will be included in the greater; it is required to find the sine and the cosine of their sum or difference, or of the angle ABD.

Construction. Through any point E, in the line BC, that is common to the two angles, draw a perpendicular FG, cutting the two other lines BA, and BD, in the points F, and G. From the point, F where this perpendicular cuts the line BA, which marks the sum or difference of these angles, let fall the perpendicular FH, upon the third line, BD.

Using the same denominations as in the primitive formulæ of series A, we make: $BG=h$; $BF=h'$; $BE=k$; $EG=d$; $EF=d'$, and calling the angle $CBD=a$; the angle $CBA=b$; and the perpendicular $FH=y$; and $BH=x$.

The $FG=d\pm d'$, will follow, with the sign $+$, for the sum, and $-$, fôr the difference, of the two angles a, and b, and we shall have the quotient or ratio:

$$\frac{y}{h'} = \sin(a \pm b); \text{ and } \frac{x}{h'} = \cos(a \pm b)$$

Solution. The triangles *FGH*, and *BGE*, are similar, being right angled at *H*, and *E*, and having the angle *G*, common to the two triangles; wherefore

(By Euclid, B. 6. Prop. 4.) $h : k = d \pm d' : y$

$$y = \frac{k\,d \pm k\,d'}{h}$$

Dividing by h',
$$\frac{y}{h'} = \frac{k\,d \pm k\,d'}{h\,h'}$$

$$= \frac{k\,d}{h'\,h} \pm \frac{k\,d'}{h\,h'}$$

$$= \frac{k}{h'} . \frac{d}{h} \pm \frac{k}{h} . \frac{d'}{h'}$$

Substituting the values of these several quotients, according to the principles of the series A, we have

F 1 $$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

For the cosine we have

(Euc. B 1. Prop. 47.) $y^2 = (d \pm d')^2 - (h - x)^2 = (h')^2 - x^2$

or $(d \pm d')^2 - h^2 + 2\,hx - x^2 = (h')^2 - x^2$

and $(d \pm d')^2 - h^2 + 2\,hx = (h')^2$

therefore $2\,hx = (h')^2 + h^2 - (d \pm d')^2$

Substituting for $h^2 = k^2 \pm d^2$; and $(h')^2 = k^2 + (d')^2$; and dividing by $2hh'$,

$$\frac{x}{h'} = \frac{2\,k^2 + (d')^2 + d^2 - (d \pm d)^2}{2hh'}$$

Squaring $(d \pm d')$ $$= \frac{2k^2 + (d')^2 + d^2 - (d') - d^2 \pm 2dd'}{2hh'}$$

By compensation $$= \frac{k\,k \mp d\,d'}{h\,h'}$$

$$\frac{x}{h'} = \frac{k}{h} \cdot \frac{k}{k'} \mp \frac{d}{h} \cdot \frac{d'}{h'}$$

Substituting for these quotients their values, according to the series A, we have

$$\cos (a \pm b) = \cos a \cos b \mp \sin a \sin b \qquad 2$$

It will be here seen, that in the result the algebraic signs of the last formulæ, are of the contrary nature to that they possess in the expression representing the sum or difference of the two angles; while in the case of the sines they have the same nature, as in the expression of the compound angle. This might also have been anticipated from the simple knowledge of the fact, that the cosine diminishes with the increase of the angle; for in every greater angle the line k, will be less, than in a less angle; while the perpendiculars increase with the increase of the angle.

§ 17. In order to find the tangent, cotangent, secant and cosecant, of the sum, or difference, of two angles; we must treat these formulæ, 1, and 2, in the same way as the simple formulæ of the series A, when those of the series B, were investigated; and then simplify them by means of these same formulæ, in conformity with what was at first said in relation to them, that they constitute the multiplication table of trigonometry, and thus furnish the means of reduction. We shall then have, (analogous to B, No. 8,)

$$\tan (a \pm b) = \frac{\sin (a \pm b)}{\cos (a \pm b)} = \frac{\sin a \cos b \pm \cos a \sin b}{\cos a \cos b \mp \sin a \sin b} \qquad 3$$

dividing this last expression in numerator and denominator, successively by the four factors contained in it, and substituting, for the resulting values, the corresponding tangents and cotangents, according to the formulæ of the series, B, we obtain in succession the following formulæ, viz.

Dividing by $\sin a \cos b$; $\tan (a \pm b) = \dfrac{1 \pm \tan b \cot a}{\cot a \mp \tan b}$ 4

5 $\sin b \cos a \qquad = \dfrac{\tan a \cot b \pm 1}{\cot b \mp \tan a}$

6 $\cos a \cos b \qquad = \dfrac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$

7 $\sin a \sin b \qquad = \dfrac{\cot b \pm \cot a}{\cot a \cot b \mp 1}$

For the value of the cotangent is obtained, analagous to B, 9.

8 $$\cot (a \pm b) = \frac{\cos (a \pm b)}{\sin (a \pm b)} = \frac{\cos a \cos b \mp \sin a \sin b}{\sin a \cos b \pm \cos a \sin b}$$

A process analogous to the preceding gives in succession the following formulæ :

9 Dividing by, $\sin a \cos b$, $\cot (a \pm b) = \dfrac{\cot a \mp \tan b}{1 \pm \tan b \cot a}$

10 $\sin b \cos a \qquad = \dfrac{\cot b \mp \tan a}{\tan a \cot b \pm 1}$

11 $\cos a \cos b \qquad = \dfrac{1 \mp \tan a \tan b}{\tan a \pm \tan b}$

12 $\sin a \sin b \qquad = \dfrac{\cot a \cot b \mp 1}{\cot b \pm \cot a}$

It may be easily seen that these formulæ for the cotangent are the inverse of those for the tangent, as might be expected from their analogy to A, No. 3, and 4.

In the same manner as before, we obtain

13 $$\sec (a \pm b) = \frac{1}{\cos (a \pm b)} = \frac{1}{\cos a \cos b \mp \sin a \sin b}$$

Dividing still in this case by the same four factors, employed in the case of the tangent, and substituting the, secants, and, cosecants, for their equals, $\frac{1}{\cos}$ & $\frac{1}{\sin}$, in conformity with the expressions of the series, B, we obtain the four following results, viz.

$$\sec(a \pm b) = \frac{\operatorname{cosec} a \sec b}{\cot a \mp \tan b} \qquad 14$$

$$\sec(a \pm b) = \frac{\sec a \operatorname{cosec} b}{\cot b \mp \tan a} \qquad 15$$

$$= \frac{\sec a \sec b}{1 \mp \tan a \tan b} \qquad 16$$

$$= \frac{\operatorname{cosec} a \operatorname{cosec} b}{\cot a \cot b \mp 1} \qquad 17$$

Substituting for the secants their values in terms of the tangents, taken from the radical expressions of series C, as has been done for series D, these formulæ undergo the following transformations, which may easily be followed without being detailed :

$$\sec(a \pm b) = \frac{(1 + \cot^2 a)^{\frac{1}{2}} (1 + \tan^2 b)^{\frac{1}{2}}}{\cot a \mp \tan b} \qquad 18$$

$$= \frac{(1 + \tan^2 a)^{\frac{1}{2}} (1 + \cot^2 b)^{\frac{1}{2}}}{\cot b \mp \tan a} \qquad 19$$

$$= \frac{(1 + \tan^2 a)^{\frac{1}{2}} (1 + \tan^2 b)^{\frac{1}{2}}}{1 \mp \tan a \tan b} \qquad 20$$

$$= \frac{(1 + \cot^2 a)^{\frac{1}{2}} (1 + \cot^2 b)^{\frac{1}{2}}}{\cot a \cot b \mp 1} \qquad 21$$

Applying a process exactly analogous to the expressions of the value of the cosecant, we obtain successively the following formulæ, which are analogous to the preceding ones :

$$\operatorname{cosec}(a \pm b) = \frac{1}{\sin(a \pm b)} = \frac{1}{\sin a \cos b \pm \cos a \sin b} \qquad 22$$

$$= \frac{\operatorname{cosec} a \sec b}{1 \pm \cot a \tan b} \qquad 23$$

24 $$\operatorname{cosec}(a \pm b = \frac{\sec a \operatorname{cosec} b}{\tan a \cot b \pm 1}$$

25 $$= \frac{\operatorname{cosec} a \operatorname{cosec} b}{\cot b \pm \cot a}$$

26 $$= \frac{\sec a \sec b}{\tan a \pm \tan b}$$

27 $$= \frac{(1 + \cot^2 a)^{\frac{1}{2}} (1 + \tan^2 b)^{\frac{1}{2}}}{1 \pm \cot a \tan b}$$

28 $$= \frac{(1 + \tan^2 a)^{\frac{1}{2}} (1 + \cot^2 b)^{\frac{1}{2}}}{\tan a \cot b \pm 1}$$

29 $$= \frac{(1 + \cot^2 a)^{\frac{1}{2}} (1 + \cot^2 b)^{\frac{1}{2}}}{\cot b \pm \cot a}$$

30 $$= \frac{(1 + \tan^2 a)^{\frac{1}{2}} (1 + \tan^2 b)^{\frac{1}{2}}}{\tan a \pm \tan b}$$

It is evident, that, if in these formulæ for secant, and cosecant, we should change the numerators into denominators, and the denominators into numerators, we should obtain expressions for the sine, and cosine; in their inverse application all these formulæ are naturally reductions of compound expressions to the simple expressions of a compound angle; if therefore we meet with such formulæ as the above in the course of a calculation, we have the means furnished us of rendering them much more simple.

CHAPTER IV.

Combinations of the Formulæ of Simple Angles.

§ 18. The use which we have made, in the last chapter of the formulæ of the series B, has given an instance of the value of the research of the combinations of trigonometric functions, as applicable to the reduction of complicated formulæ, as well as in obtaining expressions appropriate to the data that may present themselves in calculation.

As it is evident, that these combinations ought to be the result of the application of one or the other of the four rules of arithmetic, the investigation will be here made by this simple method.

It is clear, that these combinations must be very numerous; we shall therefore, in this place, rather point out the road, that leads to their discovery, than enter into a detail of all the possible combinations.

One of the frequent uses that is made of these formulæ, consists in changing an addition or subtraction into a multiplication, (in order to enable us to make use of logarithms,) and conversely. We shall therefore devote ourselves, principally, to formulæ that have properties of this sort. It will be easy, by a slight attention to the general method, to reach any other form that may be desired in any particular case.

§ 19. The simple addition and subtraction of the formula B, No. 8, applied to two angles, a, and b, assuming, $a > b$, will give G

$$\tan a \pm \tan b = \frac{\sin a}{\sin b} \pm \frac{\sin b}{\cos b} = \frac{\sin a \cos b \pm \cos a \sin b}{\cos a \cos b} \qquad 1$$

$$= \frac{\sin (a \pm b)}{\cos a \cos b}$$

2 Dividing by, $\sin a \cos b$; $\tan a \pm \tan b = \dfrac{1 \pm \cot a \tan b}{\cot a}$

3 $\cos a \sin b$; $= \dfrac{\tan a \cot b \pm 1}{\cot b}$

4 $\sin a \sin b$; $= \dfrac{\cot b \pm \cot a}{\cot a \cot b}$

From B, No. 9, treated in the same manner, we obtain

5 $\cot b \pm \cot a = \dfrac{\cos b}{\sin b} \pm \dfrac{\cos a}{\sin a} = \dfrac{\sin a \cos b \pm \cos a \sin b}{\sin a \sin b} = \dfrac{\sin (a \pm b)}{\sin a \sin b}$

6 Dividing by, $\sin a \cos b$; $\cot b \pm \cot a = \dfrac{1 \pm \cot a \tan b}{\tan b}$

7 $\cos a \sin b$; $= \dfrac{\tan a \cot b \pm 1}{\tan a}$

8 $\cos a \cos b$; $= \dfrac{\tan a \pm \tan b}{\tan a \tan b}$

From the combination of B, No. 8 & 9, applied to different angles, we obtain :

9 $\cot a \pm \tan b = \dfrac{\cos a}{\sin a} \pm \dfrac{\sin b}{\cos b} = \dfrac{\cos a \cos b \pm \sin a \sin b}{\sin a \cos b} = \dfrac{\cos (a \mp b)}{\sin a \cos b}$

10 Dividing by, $\cos a \cos b$; $\cot a \pm \tan b = \dfrac{1 \pm \tan a \tan b}{\tan a}$

11 $\sin a \sin b$; $= \dfrac{\cot a \cot b \pm 1}{\cot b}$

12 $\cos a \sin b$; $= \dfrac{\cot b \pm \tan a}{\tan a \cot b}$

and

13 $\cot b \pm \tan a = \dfrac{\cos b}{\sin b} \pm \dfrac{\sin a}{\cos a} = \dfrac{\cos a \cos b \pm \sin a \sin b}{\sin b \cos a} = \dfrac{\cos (a \mp b)}{\sin b \cos a}$

Dividing by, $\sin a \cos b$; $\cot b \pm \tan a = \dfrac{1 \pm \tan a \tan b}{\tan a}$ 14

$\sin a \sin b$; $= \dfrac{\cot a \cot b \pm 1}{\cot a}$ 15

$\cos b \sin a$; $= \dfrac{\cot a \pm \tan b}{\cot a \tan b}$ 16

The formulæ, No. 2, 3, 4; 6, 7, 8; 10, 11, 12; 14, 15, 16; might evidently have been obtained, with equal ease, by the simple multiplication or division of the sums indicated by tangent a, tangent b, or their products; but as they naturally follow, from the method that has been employed previously, and since, in this way, the different values of the sums sought are collated, it seems to be more in conformity with systematic arrangement, to present them in the way they occur above.

§ 20. The several combinations of the formulæ, 8 & 9, of series B, by means of multiplication and division, are, as is clear, contained in those which precede. In effect we have, by comparing the formulæ No. 5 & 8; No. 4 & 9; No. 9 & 12; No. 13 & 16, the following:

H

$$\tan a \tan b = \frac{\tan a}{\cot b} = \frac{\sin (a \pm b)}{\sin a \sin b (\tan a \pm \tan b)} \qquad 1$$

$$\cot a \cot b = \frac{\cot a}{\tan b} = \frac{\sin (a \pm b)}{\cos a \cos b (\cot b \pm \cot a)} \qquad 2$$

$$\tan a \cot b = \frac{\tan a}{\tan b} = \frac{\cos (a \mp b)}{\sin a \cos b (\cot b \pm \tan a)} \qquad 3$$

$$\cot a \tan b = \frac{\cot a}{\cot b} = \frac{\cos (a \mp b)}{\sin b \cos a (\cot a \pm \tan b)} \qquad 4$$

§ 21. The formulæ G, No. 1, 5, 9, & 13, are, as is evident, of such a nature as to change an addition or subtraction into a multiplication or division; they also serve, inversely, in the construction of tables to find the tangents, by means of the sines and cosines. In like manner we obtain,

by comparing: G, No. 1, 2, and 3, and No. 9, 10, and 11, the following formulæ that will be of use.

1 $$1 \pm \cot a \tan b = \frac{\sin(a \pm b) \cot a}{\cos a \cos b} = \frac{\sin(a \pm b)}{\sin a \cos b}$$

2 $$\tan a \cot b \pm 1 = \frac{\sin(a \pm b) \cot b}{\cos a \cos b} = \frac{\sin(a \pm b)}{\cos a \sin b}$$

3 $$1 \pm \tan a \tan b = \frac{\cos(a \mp b) \tan a}{\sin a \cos b} = \frac{\cos(a \mp b)}{\cos a \cos b}$$

4 $$\cot a \cot b \pm 1 = \frac{\cos(a \mp b) \cot b}{\sin a \cos b} = \frac{\cos(a \mp b)}{\cos a \cos b}$$

§ 22. By separating the signs in the formula G, No. 1, and multiplying the separate parts, we obtain a formula for the difference of the squares of the tangents, that is very simple, and analogous in its nature to the original formula; we have

5 $$(\tan a + \tan b)(\tan a - \tan b) = \tan^2 a - \tan^2 b = \frac{\sin(a+b)\sin(a-b)}{\cos^2 a \cos^2 b}$$

And similar formulæ are deduced, with equal ease, from the other formulæ of the same character; they do not however appear to require, that their investigation be given here, in detail, and they are, besides, easily found in case they are needed.

CHAPTER V.

Combination of the Formulæ of the Sum, and difference of two Angles.

§ 23. Separating the signs in the formulæ F, No. 1 and 2, and combining them, by addition and subtraction, we obtain a series of simple formulæ, that are very useful in their practical application to calculation, viz.

$\sin(a+b)+\sin(a-b)=$ K

$\sin a \cos b + \cos a \sin b + \sin a \cos b - \sin b \cos a = 2 \sin a \cos b$ 1

$\sin(a+b)-\sin(a-b)=$ 2

$\sin a \cos b + \cos a \sin b - \sin a \cos b + \sin b \cos a = 2 \cos a \sin b$

$\cos(a-b)+\cos(a+b)=$ 3

$\cos a \cos b + \sin a \sin b + \cos a \cos b - \sin a \sin b = 2 \cos a \cos b$

$\cos(a-b)-\cos(a+b)=$ 4

$\cos a \cos b + \sin a \sin b - \cos a \cos b + \sin a \sin b = 2 \sin a \sin b$

$\sin(a \pm b) \pm \cos(a \pm b) = \sin a(\cos b \mp \sin b) \pm \cos a(\cos b \pm \sin b)$ 5

As this last formula does not present any peculiar interest, it is not deduced in detail; it may be found by a simple calculation.

§ 24. The addition of the two values of F, No. 3, with their signs changed, gives the following formulæ, by means of a very simple process of reduction:

$$\tan(a \pm b) + \tan(a \mp b) = \frac{\sin(a \pm b)}{\cos(a \pm b)} + \frac{\sin(a \mp b)}{\cos(a \mp b)}$$

Reducing to a common denominator

$$= \frac{\sin(a \pm b)\cos(a \mp b) + \sin(a \mp b)\cos(a \pm b)}{\cos(a \pm b)\cos(a \mp b)}$$

The numerator being $= \sin\big((a \pm b) + (a \mp b)\big) = \sin 2a$, and performing the multiplication in the denominator, we have the above.

$$= \frac{\sin 2a}{\cos^2 a \cos^2 b - \sin^2 a \sin^2 b}$$

And because, $\cos^2 b = 1 - \sin^2 b$; and, $\sin^2 a = 1 - \cos^2 a$; and by compensation,

$$\tan(a \pm b) + \tan(a \mp b) = \frac{\sin 2a}{\cos^2 a - \sin^2 b} \qquad 6$$

The subtraction of these same two expressions, gives a result exactly similar, with this exception: that the two terms of the numerator are separated by the sign $-$, instead of

+. It results from this: that, instead of the sine of the sum of the two angles $(a \pm b)$ and $(a \mp b)$ the numerator represents the sine of the difference of these angles; we then have as numerator,

$$\sin\left((a \pm b) - (a \mp b)\right) = \sin(\pm 2\, b) = \pm \sin 2\, b$$

As the denominator does not undergo any change, the definitive formula, which requires the same steps for its reduction as the preceding, becomes

7 $$\tan(a \pm b) - \tan(a \mp b) = \frac{\pm \sin 2\, b}{\cos^2 a - \sin^2 b}$$

If we now treat in the same manner the formula for the cotangents, F, No. 8, and pay attention to the fact, that the cotangents of small angles are greater than those of large angles; and therefore, as has been already remarked, the subtraction must be inverted. We have

$$\cot(a \mp b) + \cot(a \pm b) = \frac{\cos(a \mp b)}{\sin(a \mp b)} + \frac{\cos(a \pm b)}{\sin(a \pm b)}$$

$$= \frac{\cos(a \mp b)\sin(a \pm b) + \cos(a \pm b)\sin(a \mp b)}{\sin(a \mp b)\sin(a \pm b)}$$

The numerator is evidently the same as in formula 6, and the denominator is reduced to the difference of the squares of the two terms of the formula which gives the sine of the sum or difference of two angles; we then have, again, for the angle of the numerator,

$$\sin\left((a \pm b) + (a \mp b)\right) = \sin 2a$$

And the formula will, by applying reductions to the denominator, as before, ultimately become,

8 $$\cot(a \mp b) + \cot(a \pm b) = \frac{\sin 2\, a}{\cos^2 b - \cos^2 a}$$

Subtracting the same two formulæ, we obtain, as in the case of the tangent, a numerator that represents the difference of

the angles, and consequently, has exactly the same value as in formula 7, except that the signs are inverted, in consequence of the inverted subtraction, that is to say, $(a \mp b) - (a \pm b) = \mp 2b$; and as the denominator remains the same as in formula 7, the final formula will become K

$$\cot(a \mp b) - \cot(a \pm b) = \frac{\mp \sin 2b}{\cos^2 b - \cos^2 a} \qquad 9$$

By a process precisely similar to that given above, and whose detail is omitted here, for the express purpose of giving the student an opportunity of exercise in operations of the sort, we may obtain the two following results:

$$\tan(a \pm b) + \cot(a \mp b) = \frac{\cos 2b}{\cos a \sin a \mp \sin b \cos b} \qquad 10$$

$$\tan(a \pm b) - (\cot a \mp b) = \frac{-\cos 2a}{\cos a \sin a \mp \sin b \cos b} \qquad 11$$

It is obvious, that more combinations of this sort may be made, from the corresponding formulæ.

§ 25. It will easily be seen, by inspecting the formulæ of § 23 and 24, that by dividing any one of them by any other of the corresponding formulæ, taking in § 24 those which have either the same numerator or the same denominator, we can obtain formulæ of the greatest simplicity on the one side, corresponding to expressions on the other side of the equation, that are apparently complicated. But it would be useless to make these combinations here, as they are of the greatest facility.

§ 26. The formulæ of the series F, give, by multiplication, the following results. The signs being separated, as has been done in the greater part of the formulæ of the preceding series K.

Multiplying F, No. 1, and separating the signs.

$\sin(a+b)\sin(a-b)$

$$= (\sin a \cos b + \cos a \sin b)(\sin a \cos b - \sin b \cos a)$$

$$= \sin^2 a \cos^2 b - \cos^2 a \sin^2 b$$

L And substituting, according to series C, No. 4 and 5.

1 $$\sin(a+b)\sin(a-b) = \sin^2 a - \sin^2 b$$

2 $$= \cos^2 b - \cos^2 a$$

Multiplying F, No. 2, with separation of the signs, and an analogous process.

$\cos(a+b)\cos(a-b)$

$$= (\cos a \cos b - \sin a \sin b)(\cos a \cos b + \sin a \sin b)$$

$$= \cos^2 a \cos^2 b - \sin^2 a \sin^2 b$$

3 $$= \cos^2 a - \sin^2 b$$

4 $$= \cos^2 b - \sin^2 a$$

By multiplying together, F, No. 3, separating the signs, and observing: that in conformity with B, No. 3, there is a division that always corresponds with a multiplication, because $\text{tang} = \dfrac{1}{\cot}$, we obtain the following results:

$$\tan(a+b)\tan(a-b) = \frac{\tan(a+b)}{\cot(a-b)} = \frac{\sin(a+b)\sin(a-b)}{\cos(a+b)\cos(a-b)}$$

Expressing the factors of the numerator and the denominator, multiplying them actually, and reducing, according to series C, No. 4 and 5, this formula is reduced to

5 $$\tan(a+b)\tan(a-b) = \frac{\tan(a+b)}{\cot(a-b)} = \frac{\sin^2 a - \sin^2 b}{\cos^2 b - \sin^2 a}$$

$$= \frac{\cos^2 b - \cos^2 a}{\cos^2 a - \sin^2 b}$$

We obtain, in the same manner, the two following formulæ, which are, besides, already evident from the four first formulæ of the present series.

6 $$\cot(a+b)\cot(a-b) = \frac{\cot(a+b)}{\tan(a-b)} = \frac{\cos^2 b - \sin^2 a}{\sin^2 a - \sin^2 b}$$

$$= \frac{\cos^2 a - \sin^2 b}{\cos^2 b - \sin^2 a}$$

$$\tan(a+b)\cot(a-b) = \frac{\tan(a+b)}{\tan(a-b)} = \frac{\sin a \cos a + \sin b \cos b}{\sin a \cos a - \sin b \cos b} \quad \text{L } 7$$

No. 5 and 6 evidently admit the variations in their numerator and denominator that are pointed out by the equality of the two preceding ones, No. 1 and 2, 3 and 4; and which are, besides, evident consequences of series C.

§ 27. After this explanation, the manner in which analogous formulæ for the secant and cosecant may be deduced, will be readily perceived; this introduction may, therefore, be considered as sufficient, particularly as we do not conceive it necessary to give every possible formulæ, but merely to point out an easy and systematic mode of obtaining them.

§ 28. By treating in the same manner those formulæ of the series F, which express the tangents, cotangents, secants and cosecants of the sum or difference of two angles, in terms of the tangent and cotangent of the simple angles, we might obtain a series of symetric formulæ in terms of the tangent and cotangent of the same simple angles. A great number of these are simple, and may be useful; but for the reason already stated, it will be sufficient merely to point out the method.

CHAPTER VI.

Trigonometric Functions, that express the Functions of Simple Angles, in terms of the Functions of Compound Angles.

§ 29. As it has always been assumed, in the preceding formulæ, that $a \geq b$, it being natural to make such an assumption in announcing any two quantities whose value and ratio is indeterminate, it follows from the principles of algebra, that

$$a = \tfrac{1}{2}(a+b) + \tfrac{1}{2}(a-b); \quad b = \tfrac{1}{2}(a+b) - \tfrac{1}{2}(a-b)$$

Applying these denominations to the formulæ of the series F, and limiting the investigation to the sine, cosine, and tangent, (which is sufficient to exhibit the principles of this operation, and to lead to formulæ of general application, in a short and easy manner,) we obtain in succession the following trigonometric functions, viz.

M By F No. 1 will be obtained,

$\sin a = \sin\left(\tfrac{1}{2}(a+b) + \tfrac{1}{2}(a-b)\right)$

1 $\quad = \sin\tfrac{1}{2}(a+b)\cos\tfrac{1}{2}(a-b) + \cos\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}(a-b)$

and also,

2 $\sin b = \sin\tfrac{1}{2}(a+b)\cos\tfrac{1}{2}(a-b) - \cos\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}(a-b)$

By F No. 2 will be obtained,

3 $\cos a = \cos\tfrac{1}{2}(a+b)(\cos\tfrac{1}{2}(a-b) - \sin\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}(a-b)$

and

4 $\cos b = \cos\tfrac{1}{2}(a+b)\cos\tfrac{1}{2}(a-b) + \sin\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}(a-b)$

By F No. 4 will be obtained $\tan a = \tan\left(\tfrac{1}{2}(a+b) + \tfrac{1}{2}(a-b)\right)$

5 $\quad = \dfrac{1 + \tan\tfrac{1}{2}(a-b)\cot\tfrac{1}{2}(a+b)}{\cot\tfrac{1}{2}(a+b) - \tan\tfrac{1}{2}(a-b)}$

6 and likewise $\tan b = \dfrac{1 - \tan\tfrac{1}{2}(a-b)\cot\tfrac{1}{2}(a+b)}{\cot\tfrac{1}{2}(a+b) + \tan\tfrac{1}{2}(a-b)}$

It will be at once seen, that the formulæ 5, 6, 7, of the same series, might be also employed for this purpose, and would lead to analogous results. The above formulæ, 5 and 6, naturally give the cotangent by a simple inversion.

§ 30. If we now combine these formulæ in the same manner, and by the same rules as the preceding, we shall obtain a series of formulæ that are much more simple, and are susceptible of becoming general for every proportional value of the angles.

By addition, and the simple compensation of the signs of the second terms of the sines and cosines, we obtain,

By adding No. 1 and 2, or

$$\sin a + \sin b = 2 \sin \tfrac{1}{2}(a + b) \cos \tfrac{1}{2}(a - b) \qquad 7$$

By adding No. 3 and 4, or

$$\cos a + \cos b = 2 \cos \tfrac{1}{2}(a + b) \cos \tfrac{1}{2}(a - b)$$

By adding No. 5 and 6, or $\tan a + \tan b =$ 8

$$\frac{1 + \tan \frac{1}{2}(a - b) \cot \frac{1}{2}(a + b)}{\cot \frac{1}{2}(a + b) - \tan \frac{1}{2}(a - b)} + \frac{1 - \tan \frac{1}{2}(a - b) \cot \frac{1}{2}(a + b)}{\cot \frac{1}{2}(a + b) + \tan \frac{1}{2}(a - b)}$$

And by reducing this to a common denominator and compensating:

$$\tan a + \tan b = \frac{2 \tan \frac{1}{2}(a + b)\left(1 + \tan^2 \frac{1}{2}(a - b)\right)}{1 - \tan^2 \frac{1}{2}(a - b) \tan^2 \frac{1}{2}(a + b)} \qquad 9$$

By subtraction, and a process exactly analogous to the above, we obtain;

By subtracting No. 2 from No. 1, or

$$\sin a - \sin b = 2 \sin \tfrac{1}{2}(a - b) \cos \tfrac{1}{2}(a + b) \qquad 10$$

By subtracting No. 3 from No. 4, or

$$\cos b - \cos a = 2 \sin \tfrac{1}{2}(a + b) \sin \tfrac{1}{2}(a - b) \qquad 11$$

By subtracting No. 6 from No. 5, or

$$\tan a - \tan b = \frac{2 \cot \frac{1}{2}(a - b)\left(1 + \cot^2 \frac{1}{2}(a + b)\right)}{\cot^2 \frac{1}{2}(a + b) \cot^2 \frac{1}{2}(a - b) - 1}, \qquad 12$$

We further obtain, by multiplication, as follows:

Multiplying No. 1 and 2, or, $\sin a \sin b$

$$= \sin^2 \tfrac{1}{2}(a + b) \cos^2 \tfrac{1}{2}(a - b) - \cos^2 \tfrac{1}{2}(a + b) \sin^2 \tfrac{1}{2}(a - b)$$

$$= \sin^2 \tfrac{1}{2}(a + b) - \sin^2 \tfrac{1}{2}(a - b) \qquad 13$$

$$= \cos^2 \tfrac{1}{2}(a - b) - \cos^2 \tfrac{1}{2}(a + b) \qquad 14$$

Multiplying No. 3 and 4, or, $\cos a \cos b$

$$= \cos^2 \tfrac{1}{2}(a + b) \cos^2 \tfrac{1}{2}(a - b) - \sin^2 \tfrac{1}{2}(a + b) \sin^2 \tfrac{1}{2}(a - b)$$

$$= \cos^2 \tfrac{1}{2}(a + b) - \sin^2 \tfrac{1}{2}(a - b) \qquad 15$$

$$= \cos^2 \tfrac{1}{2}(a - b) - \sin^2 \tfrac{1}{2}(a + b) \qquad 16$$

Multiplying No. 5 and 6, or

17
$$\tan a \tan b = \frac{1 - \tan^2 \frac{1}{2}(a - b) \cot^2 \frac{1}{2}(a + b)}{\cot^2 \frac{1}{2}(a + b) - \tan^2 \frac{1}{2}(a - b)}$$

For since the terms of the numerators and denominators in the two fractions are the same; being in the one a sum, in the other a difference, their product is the difference of their squares.

By the division of the similar functions of the two angles we obtain, as follows:

Dividing No. 1 by No. 2, or

$$\frac{\sin a}{\sin b} = \frac{\sin \frac{1}{2}(a + b) \cos \frac{1}{2}(a - b) + \cos \frac{1}{2}(a + b) \sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b) \cos \frac{1}{2}(a - b) - \sin \frac{1}{2}(a - b) \cos \frac{1}{2}(a + b)}$$

And dividing all the terms by the first term,

18
$$\frac{\sin a}{\sin b} = \frac{1 + \cot \frac{1}{2}(a + b) \tan \frac{1}{2}(a - b)}{1 - \cot \frac{1}{2}(a + b) \tan \frac{1}{2}(a - b)}$$

It is evident that we also have:

19
$$\frac{\sin a}{\sin b} = \frac{\tan \frac{1}{2}(a + b) \cot \frac{1}{2}(a - b) + 1}{\tan \frac{1}{2}(a + b) \cot \frac{1}{2}(a - b) - 1}$$

Dividing No. 3 by No. 4,

$$\frac{\cos a}{\cos b} = \frac{\cos \frac{1}{2}(a + b) \cos \frac{1}{2}(a - b) - \sin \frac{1}{2}(a + b) \sin \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b) \cos \frac{1}{2}(a - b) + \sin \frac{1}{2}(a + b) \sin \frac{1}{2}(a - b)}$$

And dividing by the two terms successively, as before:

20
$$\frac{\cos a}{\cos b} = \frac{1 - \tan \frac{1}{2}(a + b) \tan \frac{1}{2}(a - b)}{1 + \tan \frac{1}{2}(a + b) \tan \frac{1}{2}(a - b)}$$

21
$$= \frac{\cot \frac{1}{2}(a + b) \cot \frac{1}{2}(a - b) - 1}{\cot \frac{1}{2}(a + b) \cot \frac{1}{2}(a - b) + 1}$$

The division of the *sine* and *cosine* of the same angle, evidently gives the formulæ 5 and 6, and cross divisions give analogous formulæ, expressed in terms of *tangent* and *cotangent*, which may be readily found when needed.

Dividing the *tangents*, we have by the formulæ 5 and 6,

$$\frac{\tan a}{\tan b}=\frac{\left(1+\tan\frac{1}{2}(a-b)\cot\frac{1}{2}(a+b)\right)\left(\cot\frac{1}{2}(a+b)+\tan\frac{1}{2}(a-b)\right)}{\left(1-\tan\frac{1}{2}(a-b)\cot\frac{1}{2}(a+b)\right)\left(\cot\frac{1}{2}(a+b)-\tan\frac{1}{2}(a-b)\right)}$$

$$=\frac{\cot\frac{1}{2}(a+b)+\tan\frac{1}{2}(a-b)+\tan\frac{1}{2}(a-b)\cot^2\frac{1}{2}(a+b)+\cot\frac{1}{2}(a+b)\tan^2\frac{1}{2}(a-b)}{\cot\frac{1}{2}(a+b)-\tan\frac{1}{2}(a-b)-\tan\frac{1}{2}(a-b)\cot^2\frac{1}{2}(a+b)+\cot\frac{1}{2}(a+b)\tan^2\frac{1}{2}(a-b)}$$

$$=\frac{\cot\frac{1}{2}(a+b)\left(1+\tan^2\frac{1}{2}(a-b)\right)+\tan\frac{1}{2}(a-b)\left(1+\cot^2\frac{1}{2}(a+b)\right)}{\cot\frac{1}{2}(a+b)\left(1+\tan^2\frac{1}{2}(a-b)\right)-\tan\frac{1}{2}(a-b)\left(1+\cot^2\frac{1}{2}(a+b)\right)}$$

And from series C, No. 6 and 7,

$$=\frac{\cot\frac{1}{2}(a+b)\sec^2\frac{1}{2}(a-b)+\tan\frac{1}{2}(a-b)\operatorname{cosec}^2\frac{1}{2}(a+b)}{\cot\frac{1}{2}(a+b)\sec^2\frac{1}{2}(a-b)-\tan\frac{1}{2}(a-b)\operatorname{cosec}^2\frac{1}{2}(a+b)}$$

Taking from series B, No. 1 and 2, squared; that is to say, $\sec^2=\dfrac{1}{\cos^2}$; $\operatorname{cosec}^2=\dfrac{1}{\sin^2}$; which values being introduced in the formula, and the terms reduced to a common denominator, that is compensated in the numerator and denominator, the formula is reduced to the following, viz:

$$=\frac{\cot\frac{1}{2}(a+b)\sin^2\frac{1}{2}(a+b)+\tan\frac{1}{2}(a-b)\cos^2\frac{1}{2}(a-b)}{\cot\frac{1}{2}(a+b)\sin^2\frac{1}{2}(a+b)-\tan\frac{1}{2}(a-b)\cos^2\frac{1}{2}(a-b)}$$

Which by compensation is finally reduced to

$$\frac{\tan a}{\tan b}=\frac{\sin\frac{1}{2}(a+b)\cos\frac{1}{2}(a+b)+\sin\frac{1}{2}(a-b)\cos\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)\cos\frac{1}{2}(a+b)-\sin\frac{1}{2}(a-b)\cos\frac{1}{2}(a-b)} \quad 22$$

a formula that is rather curious than useful, and analogous to L, No. 7.

§ 31 The preceding formulæ from No. 7 may be combined for different uses; but as we may now assume the student to possess a sufficient knowledge of this method of deducing compound trigonometric functions, we shall only mention a

few obtained by division, that are of such frequent use, that it would be improper to omit them.

N Dividing No. 7 by No. 8, or

1 $$\frac{\sin a + \sin b}{\cos a + \cos b} = \frac{2 \sin \frac{1}{2}(a + b) \cos \frac{1}{2}(a - b)}{2 \cos \frac{1}{2}(a + b) \cos \frac{1}{2}(a - b)} = \tan \tfrac{1}{2}(a + b)$$

Dividing No. 10 by No. 11, or

2 $$\frac{\sin a - \sin b}{\cos b - \cos a} = \frac{2 \sin \frac{1}{2}(a - b) \cos \frac{1}{2}(a + b)}{2 \sin \frac{1}{2}(a + b) \sin \frac{1}{2}(a - b)} = \cot \tfrac{1}{2}(a + b)$$

Dividing No. 7 by No. 10, or

3 $$\frac{\sin a + \sin b}{\sin a - \sin b} = \frac{2 \sin \frac{1}{2}(a + b) \cos \frac{1}{2}(a - b)}{2 \sin \frac{1}{2}(a - b) \cos \frac{1}{2}(a + b)} = \frac{\tan \frac{1}{2}(a + b)}{\tan \frac{1}{2}(a - b)}$$

Dividing No. 8 by No. 11, or

4 $$\frac{\cos a + \cos b}{\cos b - \cos a} = \frac{2 \cos \frac{1}{2}(a + b) \cos \frac{1}{2}(a - b)}{2 \sin \frac{1}{2}(a + b) \sin \frac{1}{2}(a - b)} = \frac{\cot \frac{1}{2}(a + b)}{\tan \frac{1}{2}(a - b)}$$

Dividing No. 7 by No. 11, or

5 $$\frac{\sin a + \sin b}{\cos b - \cos a} = \frac{2 \sin \frac{1}{2}(a + b) \cos \frac{1}{2}(a - b)}{2 \sin \frac{1}{2}(a + b) \sin \frac{1}{2}(a - b)} = \cot \tfrac{1}{2}(a - b)$$

Dividing No. 10 by No. 8, or

6 $$\frac{\sin a - \sin b}{\cos b + \cos a} = \frac{2 \sin \frac{1}{2}(a - b) \cos \frac{1}{2}(a + b)}{2 \cos \frac{1}{2}(a - b) \cos \frac{1}{2}(a + b)} = \tan \tfrac{1}{2}(a - b)$$

CHAPTER VII.

General Formulæ for the Multiples of Angles.

§ 32 The formulæ of the preceding chapter, series M, are susceptible of the utmost generalization, and give, in this way, general values of the trigonometric functions of the multiples of angles, expressed in terms of the functions of the simple

angle or some inferior multiple. To do this we need only assign to a, and b, a certain relative value, expressed in general terms; we may then transcribe the formulæ in the form they assume under this supposition, transpose and reduce them by known compensations.

Let us assume for the two angles, a, and b, the following proportional multiples: of a, make $n\,a$; and of b, make $(n-2)\,a$.

Taking the formulæ 7, 8, 9, 10, 11, 12; transposing in the first place, in order to abridge the operation, all the second terms of each equation; we have as follows:

O

From M,

No. 7; $\sin na = 2\sin(n-1)\,a\cos a - \sin(n-2)\,a$ 1

8; $\cos na = 2\cos(n-1)\,a\cos a - \cos(n-2)\,a$ 2

$$9;\ \tan na = \frac{2\tan(n-1)\,a\,(1+\tan^2 a)}{1-\tan^2(n-1)\,a\tan^2 a} - \tan(n-2)\,a =$$

$$\frac{2\tan(n-1)\,a\sec^2 a - \tan(n-2)\,a\left(1-\tan^2(n-1)\,a\tan^2 a\right)}{1-\tan^2(n-1\,a\tan^2 a} \qquad 3$$

From M,

No. 10; $\sin na = 2\cos(n-1)\,a\sin a + \sin(n-2)\,a$ 4

11; $\cos na = -2\sin(n-1)\,a\sin a + \cos(n-2)\,a$ 5

$$12;\ \tan na = \frac{2\cot a\left(1+\cot^2(n-1)\,a\right)}{\cot^2(n-1)\,a\cot^2 a - 1} - \frac{1}{\cot(n-2)\,a} =$$

$$\frac{2\cot a\,\mathrm{cosec}^2(n-1)\,a\cot(n-2)\,a - \cot^2(n-1)\,a\cot^2 a + 1}{\cot(n-2)\,a\left(\cot^2(n-1)\,a - \cot^2 a - 1\right)} \qquad 6$$

Notwithstanding the complicated appearance of No. 3 and 6, they may be reduced to forms comparatively simple in their application to numbers substituted for n.

We moreover have, by the formulæ 5 and 6 of series M, expressions that fulfil (though in part only) the same object.

O By transforming

7 **M** No. 5; $\tan na = \dfrac{1 + \tan a \tan (n-1) a}{\cot (n-1) a - \tan a}$

8 6; $\tan (n-2) a = \dfrac{1 - \tan a \cot (n-1) a}{\cot (n-1) a + \tan a}$

§ 33 If we give to n, the value of the several terms of the series of natural numbers in succession, we may obtain from the preceding formulæ two series of expressions for multiple angles in a regular ascending order. Thus we have, from the formulæ O, No. 1 and 4, by successive assumptions of the

P value of n, $= 1, 2, 3,$ &c.

$$\begin{cases} \sin a = \sin a & = \sin a \\ \sin 2a = 2 \sin a \cos a & = 2 \cos a \sin a \\ \sin 3a = 2 \sin 2a \cos a - \sin a & = 2 \cos 2a \sin a + \sin a \\ \sin 4a = 2 \sin 3a \cos a - \sin 2a & = 2 \cos 3a \sin a + \sin 2a \\ \sin 5a = 2 \sin 4a \cos a - \sin 3a & = 2 \cos 4a \sin a + \sin 3a \\ \sin 6a = 2 \sin 5a \cos a - \sin 4a & = 2 \sin 5a \sin a + \sin 4a \\ \sin 7a = \&c. & \end{cases}$$

1

2

From the formulæ 2 and 5, we obtain the following series for the cosines:

$$\begin{cases} \cos a = \cos a & = \cos a \\ \cos 2a = \cos^2 a - 1 & = - 2 \sin^2 a + 1 \\ \cos 3a = 2 \cos 2a \cos a - \cos a & = - 2 \sin 2a \sin a + \cos a \\ \cos 4a = 2 \cos 3a \cos a - \cos 2a & = - 2 \sin 3a \sin a + \cos 2a \\ \cos 5a = 2 \cos 4a \cos a - \cos 3a & = - 2 \sin 4a \sin a + \cos 3a \\ \cos 6a = 2 \cos 5a \cos a - \cos 4a & = - 2 \sin 5a \sin a + \cos 4a \\ \cos 7a = \&c. & \end{cases}$$

3

4

We obtain for the tangents from the formula O, No. 3,

$$\left.\begin{aligned}
\tan a &= \tan a \\
\tan 2a &= \frac{2\tan a}{1-\tan^2 a} \\
\tan 3a &= \frac{2\tan 2a + 2\tan 2a\tan^2 a - \tan a + \tan^2 2a\tan^3 a}{1-\tan^2 2a\tan^2 a} \\
\tan 4a &= \frac{2\tan 3a + 2\tan 3a\tan^2 a - \tan 2a + \tan^2 3a\tan 2a\tan^2 a}{1-\tan^2 3a\tan^2 a} \\
\tan 5a &= \frac{2\tan 4a + 2\tan 4a\tan^2 a - \tan 3a + \tan^2 4a\tan 3a\tan^2 a}{1-\tan^2 4a\tan^2 a} \\
\tan 6a &= \frac{2\tan 5a + 2\tan 5a\tan^2 a - \tan 4a + \tan^2 5a\tan 4a\tan^2 a}{1-\tan^2 5a\tan^2 a} \\
\tan 7a &= \&c.
\end{aligned}\right\} 5$$

From the formula O, No. 6, we have

$$\left.\begin{aligned}
\tan a &= \frac{1}{\cot a} \\
\tan 2a &= \frac{2\cot a}{\cot^2 a - 1} \\
\tan 3a &= \frac{2\cot^2 a + \cot^2 a\cot^2 2a + 1}{\cot^3 a\cot^2 2a - \cot a} \\
\tan 4a &= \frac{2\cot a\cot 2a + 2\cot a\cot 2a\cot^2 3a - \cot^2 a\cot^2 3a + 1}{\cot 2a\cot^2 3a\cot^2 a - \cot 2a} \\
\tan 5a &= \frac{2\cot a\cot 3a + 2\cot a\cot 3a\cot^2 4a - \cot^2 a\cot^2 4a + 1}{\cot 3a\cot^2 4a\cot^2 a - \cot 3a} \\
\tan 6a &= \frac{2\cot a\cot 4a + 2\cot a\cot 4a\cot^2 5a - \cot^2 a\cot^2 5a + 1}{\cot 4a\cot^2 5a\cot^2 a - \cot 4a} \\
\tan 7a &= \&c.
\end{aligned}\right\} 6$$

All these formulæ may be changed into such as have no other functions involved, than those of the simple angle; by sub-

stituting successively in the formulæ of the multiple angles, the values found for their different factors, in the expressions that precede them. There will result, as may be foreseen, a double series of formulæ, which will follow regular laws. From these may be deduced, finally, a general law for the combination of the trigonometric functions of multiple angles, in terms of the simple angle. It may readily be conceived, that, in consequence of the multiplicity of formulæ furnished by the trigonometric functions, many similar series may be made, varied in a high degree, and adapted to every varying purpose. It will be also readily seen, that such series must finally lead to a general law, in the same way that the binomial theorem furnishes the law that governs the combination of the different powers of two quantities.

CHAPTER VIII.

Formulæ for Double Angles, expressed in terms of the Functions of the Simple Angles, and for Half Angles, expressed in terms of the Functions of the Whole Angle.

§ 34. Among the formulæ that express the functions of multiple angles, in terms of the functions of the simple angle; those which give the functions of the double angle, in terms of the functions of the simple angle; and the functions of the half angle, in terms of the whole angle; are of such frequent use, that it is proper to treat of them separately, and to collect the results for future use.

In order to obtain them, it is sufficient to assume, in the formulæ of the series F, with the sign $+$, the value of $a = b$, whence $a + b = 2\,a$; to transcribe them here with the reductions produced by the calculation itself, and the formulæ of the series B and C. In order to abridge the work, and present at one view these formulæ in a short table, by which

the frequent use that will be made of them may be facilitated, we shall suppose that recourse is had to these series for the explanation of the requisite operations, and that it is not necessary to refer to them in every particular instance.

			Q
By F No. 1, is obtained;	$\sin 2a$	$= 2 \sin a \cos a$	1
2,	$\cos 2a$	$= \cos^2 a - \sin^2 a$	2
		$= 1 - 2 \sin^2 a$	3
		$= 2 \cos^2 a - 1$	4
4,	$\tan 2a$	$= \dfrac{2}{\cot a - \tan a}$	5
6,		$= \dfrac{2 \tan a}{1 - \tan^2 a}$	6
7,		$= \dfrac{2 \cot a}{\cot^2 a - 1}$	7
8,	$\cot 2a$	$= \dfrac{\cot a - \tan a}{2}$	8
11,		$= \dfrac{1 - \tan^2 a}{2 \tan a}$	9
12,		$= \dfrac{\cot^2 a - 1}{2 \cot a}$	10

The three last formulæ being evidently a mere inversion of the three foregoing, as might be expected.

By F, No. 13, is obtained;	$\sec 2a$	$= \dfrac{1}{\cos^2 a - \sin^2 a}$	11
and hence,		$= \dfrac{1}{2 \cos^2 a - 1}$	12
		$= \dfrac{1}{1 - 2 \sin^2 a}$	13

Q

14 By F, No. 16, $\sec 2a = \frac{\sec^2 a}{1 - \tan^2 a}$,

15 17, $= \frac{\mathrm{cosec}^2 a}{\cot^2 a - 1}$

16 20, $= \frac{1 + \tan^2 a}{1 - \tan^2 a}$

17 21, $= \frac{1 + \cot^2 a}{\cot^2 a - 1}$

18 22, $\mathrm{cosec}\, 2a = \frac{1}{2 \sin a \cos a}$

19 25, $= \frac{\mathrm{cosec}^2 a}{2 \cot a}$

20 26, $= \frac{\sec^2 a}{2 \tan a}$

21 29, $= \frac{1 + \cot^2 a}{2 \cot a}$

22 30, $= \frac{1 + \tan^2 a}{2 \tan a}$

The formulæ for the secant give new formulæ for the cosine, by changing the numerator into denominator, and inversely. The formulæ for the cosecant give new formulæ for the sine, by the same process.

The formulæ F, No. 5, and 6; 18, and 19; 27, and 28, have not been employed, because they furnish less simple formulæ, and it has not been considered necessary to swell this table by using them.

§ 35. For the expression of the sine of half the angle, in terms of the functions of the whole angle, we first obtain, by the transposition of the formula Q, No. 3, writing $\frac{1}{2} a$, instead of a; and a, instead of $2 a$.

$$2 \sin^2 \tfrac{1}{2} a = 1 - \cos a \qquad \text{whence} \qquad \text{R}$$

$$\sin \tfrac{1}{2} a = \frac{(1 - \cos a)^{\frac{1}{2}}}{\sqrt{2}} \qquad 1$$

Substituting, in the first of these formulæ, the values taken from C, No. 5, we have,

$$2 \sin^2 \tfrac{1}{2} a = 1 - (1 - \sin^2 a)^{\frac{1}{2}}$$

The part under the radicals, may be considered as the difference of two squares, and expressed by the product of the sum and difference of its roots. This transforms it into

$$2 \sin^2 \tfrac{1}{2} a = 1 - (1 + \sin a)^{\frac{1}{2}} (1 - \sin a)^{\frac{1}{2}}$$

Adding, on the right hand side of the equation,

$\dfrac{\sin a}{2} - \dfrac{\sin a}{2} = o$, which does not change its value, and making $1 = \frac{1}{2} + \frac{1}{2}$, it becomes

$$2 \sin^2 \tfrac{1}{2} a = \tfrac{1}{2} + \frac{\sin a}{2} + \tfrac{1}{2} - \frac{\sin a}{2} - (1 + \sin a)^{\frac{1}{2}} (1 - \sin a)^{\frac{1}{2}}$$

$$= \frac{1 - \sin a}{2} + \frac{1 - \sin a}{2} - (1 + \sin a)^{\frac{1}{2}} (1 - \sin a)^{\frac{1}{2}}$$

As this forms a complete square, we may extract the root, which gives us

$$\sin \tfrac{1}{2} a \sqrt{2} = \frac{(1 + \sin a)^{\frac{1}{2}}}{\sqrt{2}} - \frac{(1 - \sin a)^{\frac{1}{2}}}{\sqrt{2}}$$

and dividing by $\sqrt{2}$,

$$\sin \tfrac{1}{2} a = \tfrac{1}{2} (1 + \sin a)^{\frac{1}{2}} - \tfrac{1}{2} (1 - \sin a)^{\frac{1}{2}} \qquad 2$$

We obtain for the expression of the cosine, by taking the formula F, No. 4, and treating it in exactly the same manner that we have done for the sine, the following results, in succession; viz.

$$2 \cos^2 \tfrac{1}{2} a = 1 + \cos a$$

R whence

3 $$\cos \tfrac{1}{2} a = \frac{(1 + \cos a)^{\frac{1}{2}}}{\sqrt{2}}$$

And for the formula analogous to No. 2, of this series,

$$2 \cos^2 \tfrac{1}{2} a = 1 + (1 - \sin^2 a)^{\frac{1}{2}}$$

$$= 1 + (1 + \sin a)^{\frac{1}{2}} (1 - \sin a)^{\frac{1}{2}}$$

$$= \frac{1 + \sin a}{2} + \frac{1 - \sin a}{2} + (1 + \sin a)^{\frac{1}{2}} (1 - \sin a)^{\frac{1}{2}}$$

$$\cos \tfrac{1}{2} a \sqrt{2} = \frac{(1 + \sin a)^{\frac{1}{2}}}{\sqrt{2}} + \frac{(1 - \sin a)^{\frac{1}{2}}}{\sqrt{2}}$$

4 $$\cos \tfrac{1}{2} a = \tfrac{1}{2} (1 + \sin a)^{\frac{1}{2}} + \tfrac{1}{2} (1 - \sin a)^{\frac{1}{2}}$$

We obtain an expression for the tangent, in a very simple way, by dividing the expression for the sine by that for the cosine, thus:

5 $$\tan \tfrac{1}{2} a = \frac{\sin \tfrac{1}{2} a}{\cos \tfrac{1}{2} a} = \frac{(1 - \cos a)^{\frac{1}{2}}}{(1 + \cos a)^{\frac{1}{2}}}$$

Multiplying both numerator and denominator by $(1 - \cos a)^{\frac{1}{2}}$ we obtain

6 $$\tan \tfrac{1}{2} a = \frac{1 - \cos a}{(1 + \cos a)^{\frac{1}{2}} (1 - \cos a)^{\frac{1}{2}}} = \frac{1 - \cos a}{(1 - \cos^2 a)^{\frac{1}{2}}} = \frac{1 - \cos a}{\sin a}$$

Or, multiplying in the same way by $(1 + \cos a)^{\frac{1}{2}}$

7 $$\tan \tfrac{1}{2} a = \frac{(1 - \cos a)^{\frac{1}{2}} (1 + \cos a)^{\frac{1}{2}}}{1 + \cos a} = \frac{(1 - \cos^2 a)^{\frac{1}{2}}}{1 + \cos a} = \frac{\sin a}{1 + \cos a}$$

By the division of No. 2 of this series by No. 4, we obtain:

8 $$\tan \tfrac{1}{2} a = \frac{(1 + \sin a)^{\frac{1}{2}} - (1 - \sin a)^{\frac{1}{2}}}{(1 + \sin a)^{\frac{1}{2}} + (1 - \sin a)^{\frac{1}{2}}} = \frac{1 + \sin a - \cos a}{1 + \sin a + \cos a}$$

CHAPTER IX.

Trigonometric Functions of Compound Angles, of which one Part has a Determinate Value.

§ 36. In the formulæ of the series F, whence we have deduced the elementary formulæ for angles in a *determinate ratio* to each other, we may also assume: that one of the angles has a *determinate value;* and deduce useful formulæ.

In this case it will be proper to make use of angles of which the trigonometric functions that are to be employed are simple quantities. Taking then, as in series E, $h = 1$, and comparing the values of d and k, for the several values of the angles; keeping also in mind the principles that have been already explained, namely, that

$\sin 45° = \cos 45°$, and $\tan 45° = \cot 45°$; $\sec 45° = \operatorname{cosec} 45°$;

it will be found, by the application of the formula,

$$h^2 = d^2 + k^2,$$

S

that $$\sin 45° = \cos 45° = \frac{1}{\sqrt{2}} \qquad 1$$

$$\tan 45° = \cot 45° = 1 \qquad 2$$

$$\sec 45° = \operatorname{cosec} 45° = \sqrt{2} \qquad 3$$

It being a property of the circle, that the chord of the sixth part of the circumference is equal to radius; and that the perpendicular drawn from the centre upon this chord divides it into two equal parts, which represent the d, of the half of this sixth part of the circumference; that is to say, that as the sixth part of the circle is 60°, these parts represent each the value of d, for an angle of 30°; we have in addition to the above,

$$\sin 30° = \cos 60° = \tfrac{1}{2} \qquad 4$$

And because $\sin^2 a = 1 - \cos^2 a$

S

5 $$\cos 30^\circ = \sin 60^\circ = \left(1 - \left(\tfrac{1}{2}\right)^2\right)^{\frac{1}{2}} = \frac{\sqrt{3}}{2}$$

6 $$\tan 30^\circ = \cot 60^\circ = \frac{1}{\sqrt{3}}$$

7 $$\cot 30^\circ = \tan 60^\circ = \sqrt{3}$$

8 $$\sec 30^\circ = \operatorname{cosec} 60^\circ = \frac{2}{\sqrt{3}}$$

9 $$\operatorname{cosec} 30^\circ = \sec 60^\circ = 2$$

There are a number of other similar values, to be found in the trigonometric functions, the research of which is called Rational Trigonometry. But we have no room to inquire into these, in this treatise.

§ 37. The formulæ that employ these determinate angles, are easily deduced from those of the series F.

Assuming, in the first place, that $a = 45°$, we have

10 By F, No. 1; $$\sin(45^\circ \pm b) = \cos(45^\circ \mp b) = \frac{\cos b \pm \sin b}{\sqrt{2}}$$

11 2; $$\cos(45^\circ \pm b) = \sin(45^\circ \mp b) = \frac{\cos b \mp \sin b}{\sqrt{2}}$$

12 3; $$\tan(45^\circ \pm b) = \cot(45^\circ \mp b) = \frac{\cos b \pm \sin b}{\cos b \mp \sin b}$$

13 4; $$= \quad = \frac{1 \pm \tan b}{1 \mp \tan b}$$

14 5; $$= \quad = \frac{\cot b \pm 1}{\cot b \mp 1}$$

These formulæ also give those of the other trigonometric functions, by their inversion, which therefore are not repeated here.

Substituting these values in succession, in the formulæ of the series K, dividing by $\sqrt{2}$ in the two first, and reducing

the two last, as indicated in the formulæ Q, No. 3 and 4, we have:

By K, No. 1 and 3;

$$\frac{\sin(45^\circ + b) + \sin(45^\circ - b)}{\sqrt{2}} = \cos b = \frac{\cos(45^\circ + b) + \cos(45^\circ - b)}{\sqrt{2}} \quad 15$$

By K, No. 2 and 4;

$$\frac{\sin(45^\circ + b) - \sin(45^\circ - b)}{\sqrt{2}} = \sin b = \frac{\cos(45^\circ - b) - \cos(45^\circ + b)}{\sqrt{2}} \quad 16$$

By K, No. 6 and 8;

$$\tan(45^\circ + b) + \tan(45^\circ - b) = \frac{2}{\cos 2b} = \cot(45^\circ - b) + \cot(45^\circ + b) \quad 17$$

By K, No. 7 and 9;

$$\tan(45^\circ + b) - \tan(45^\circ - b) = \frac{2 \sin 2b}{\cos 2b} = \cot(45^\circ - b) - \cot(45^\circ + b) \quad 18$$

If we substitute, in these same formulæ, the values of the sines, cosines, tangents, and cotangents, of the angles of 60°, and 30°, ascribing these values to the angle a, we obtain the following formulæ:

By F, No. 1; $\sin(30^\circ \pm b) = \dfrac{\cos b \pm \sin b \sqrt{3}}{2} = \sin(60^\circ \pm b)$ 19

2; $\cos(30^\circ \pm b) = \dfrac{\cos b \sqrt{3} \mp \sin b}{2} = \sin(60^\circ \pm b)$ 20

3; $\tan(30^\circ \pm b) = \dfrac{\cos b \pm \sin b \sqrt{3}}{\cos b \sqrt{3} \mp \sin b} = \cot(60^\circ \pm b)$ 21

4; $= \dfrac{1 \pm \tan b \sqrt{3}}{\sqrt{3} \mp \tan b}$ 22

5; $= \dfrac{\cot b \pm \sqrt{3}}{\cot b \sqrt{3} \mp 1}$ 23

H

24 By K, No. 1 ; $\cos b = \sin(30^\circ + b) + \sin(30^\circ - b)$
$= \cos(60^\circ + b) + \cos(60^\circ - b)$

25 2 ; $\sin b\sqrt{3} = \sin(30^\circ + b) - \sin(30^\circ - b)$
$= \cos(60^\circ - b) - \cos(60^\circ + b)$

26 3 ; $\cos b\sqrt{3} = \cos(30^\circ + b) + \cos(30^\circ - b)$
$= \sin(60^\circ + b) + \sin(60^\circ - b)$;

27 4 ; $\sin b = \cos(30^\circ - b) - \cos(30^\circ + b)$
$= \sin(60^\circ + b) - \sin(60^\circ - b)$

28 6 ; $\dfrac{2\sqrt{3}}{3 - 4\sin^2 b} = \tan(30^\circ \pm b) + \tan(30^\circ \mp b)$

29 7 ; $\dfrac{4\sin 2b}{3 - 4\sin^2 b} = \tan(30^\circ + b) - \tan(30^\circ - b)$

30 8 ; $\dfrac{2\sqrt{3}}{4\cos^2 b - 3} = \cot(30^\circ \mp b) + \cot(30^\circ \pm b)$

31 9 ; $\dfrac{4\sin 2b}{4\cos^2 b - 3} = \cot(30^\circ - b) - \cot(30^\circ + b)$

32 6 ; $\dfrac{\sqrt{3}}{\cos 2b} = \tan(60^\circ \pm b) + \tan(60^\circ \mp b)$

33 7 ; $\dfrac{4\sin 2b}{1 - 4\sin^2 b} = \tan(60^\circ \pm b) - \tan(60^\circ \mp b)$

34 8 ; $\dfrac{2\sqrt{3}}{4\cos^2 b - 1} = \cot(60^\circ \mp b) + \cot(60^\circ \pm b)$

35 9 ; $\dfrac{4\sin 2b}{4\cos^2 b - 1} = \cot(60^\circ \mp b) - \cot(60^\circ \pm b)$

These formulæ will be more than sufficient to show the manner in which this investigation is performed. They are, besides, of little use at present, although they were employed, at least in part, in the first construction of trigonometric tables, before they were expressed in an analytic form.

CHAPTER X.

Elementary Considerations in relation to the Application of Trigonometric Functions to Analysis, and to Calculations in general.

§ 38. It has been seen, by the series E, that the several trigonometric functions assume, in succession, every value between 0, and Infinity, both with the positive and negative sign. They are, in consequence, capable of representing every possible quantity that can occur in calculation, and the different combinations of the trigonometric functions give the same combinations of these quantities, that they do of the trigonometric functions themselves.

The inspection of several of these formulæ has already shown, that they may, for instance, serve to change an addition or subtraction into a multiplication or division; and thus transform a calculation by natural numbers, into one by logarithms; and in like manner to produce other changes of the form of calculations. Tables have even been made to facilitate such calculations. In the course of the solutions of triangles, whether plane or spheric, a frequent use will be made of them, by means of what are called auxiliary angles; and since the method is the same for all other quantities, it will be sufficiently illustrated by this application of it.

§ 39. There is another case where these functions are of great and general use in analysis. It deserves particular consideration, in consequence of the nature of the changes it demands in the trigonometric formulæ, and of its great utility. We speak of its application to those quantities that are called transcendental, which are reducible to circular arcs, and occur frequently in the integral calculus.

In order to prepare the way for this application of our formulæ, it must be, in the first place, observed: that if

x = sine a, we have a = arc whose sine is x. This is commonly expressed thus:

$$a = \text{arc} : \sin (x)$$

And so in all other cases; this expression is nothing more than the algebraic mode of expressing the idea; as in the case of sine, cosine, &c. If, then, we assume:

$$x = \sin a$$
$$y = \cos a$$
$$z = \tan a$$

we obtain for the elementary expressions of this mode of notation the following formulæ, which are analogous to those of series A in the beginning, viz:

1 $$a = \text{arc} : \sin (x)$$

2 $$= \text{arc} : \cos (y)$$

3 $$= \text{arc} : \tan (z)$$

4 $$= \text{arc} : \cot \left(\frac{1}{z}\right)$$

5 $$= \text{arc} : \sec \left(\frac{1}{y}\right)$$

6 $$= \text{arc} : \text{cosec} \left(\frac{1}{x}\right)$$

7 $$= \text{arc} : \tan \left(\frac{x}{y}\right)$$

8 $$= \text{arc} : \cot \left(\frac{y}{x}\right)$$

§ 40. We have also, by analogy with series C, the following properties of these same quantities, with all their possible algebraic variations; viz:

$$x^2 + y^2 = 1$$

$$1 + z^2 = \frac{1}{y^2}$$

$$1 + \frac{1}{z^2} = \frac{1}{x^2}$$

Whence may be deduced, as in the preceding section, the following formulæ :

$$a = \text{arc} : \sin\left(\sqrt{(1 - y^2)}\right) \quad 9$$

$$= \text{arc} : \cos\left(\sqrt{(1 - x^2)}\right) \quad 10$$

$$= \text{arc} : \tan\left(\frac{\sqrt{(1 - y^2)}}{y}\right) = \text{arc} : \tan\left(\frac{x}{\sqrt{(1 - x^2)}}\right) \quad 11$$

$$= \text{arc} : \cot\left(\frac{\sqrt{(1 - x^2)}}{x}\right) = \text{arc} : \cot\left(\frac{y}{\sqrt{(1 - y^2)}}\right) \quad 12$$

$$= \text{arc} : \sec\left(\frac{1}{\sqrt{(1 - x^2)}}\right) = \text{arc} : \sec\left(\sqrt{(1 + z^2)}\right) \quad 13$$

$$= \text{arc} : \text{cosec}\left(\frac{1}{\sqrt{(1 - y^2)}}\right) = \text{arc} : \text{cosec}\left(\frac{\sqrt{(1 + z^2)}}{z}\right) \quad 14$$

By means of these formulæ, therefore, we may express, by one or more arcs, any quantity that is given in the form of one of the trigonometric formulæ. Considering this quantity as representing the corresponding trigonometric function, the arc (here called a) will be that which is denoted by the trigonometric function, to which this quantity has been assumed to be equal, in conformity with the fundamental denominations assumed.

It must be also observed; that the simple function that is to be represented, must be that which corresponds to the function involved in the complex formula; since the object of this kind of transformation is always that of disengaging this quantity from its complications, by means of the relations of the several trigonometric functions; as in this example :

$$a = \text{arc} : \sin(x) = \text{arc} : \cot\left(\frac{\sqrt{(1 - x^2)}}{x}\right)$$

$$
\begin{aligned}
a &= \text{arc} : \cos\left(\sqrt{(1 - x^2)}\right) \\
&= \text{arc} : \tan\left(\frac{x}{\sqrt{(1 - z^2)}}\right) \\
&= \text{arc} : \sec\left(\frac{1}{\sqrt{(1 - x^2)}}\right)
\end{aligned}
$$

and in like manner in every other case.

§ 41. To attain the object of this chapter, it will be sufficient to apply this mode of transforming functions to the fundamental formulæ of the series F and Q. These occûr most frequently, and will show the manner of applying this method to every other form furnished by Analytic Trigonometry.

For the series F, we must also assume another arc $= b$; of which the functions, corresponding to those of the arc a, are best distinguished merely by an accent; so that we have for the two arcs a, and b,

$$
\begin{aligned}
x &= \sin a \qquad & x' &= \sin b \\
y &= \cos a \qquad & y' &= \cos b \\
z &= \tan a \qquad & z' &= \tan b
\end{aligned}
$$

with all their consequences, as above explained.

Substituting these values, the formula F, No. 1, will be represented in the following manner.

$$
\begin{aligned}
\sin(a \pm b) = x y' \pm x' y &= x\sqrt{(1 - x'^2)} \pm x' \sqrt{(1 - x^2)} \\
&= y' \sqrt{(1 - y^2)} \pm y \sqrt{(1 - y'^2)}
\end{aligned}
$$

Which will give, according to the principles that have been laid down, values such as:

$$
\sin\left(\text{arc} : \sin(x) \pm \text{arc} : \sin(x')\right) = x \sqrt{\left(1 - (x')^2\right)} \pm x' \sqrt{(1 - x^2)}
$$

and all the similar ones, that may be drawn from the preceding expressions.

If we take, on both sides of the equations, the arcs whose trigonometric functions are represented by these quantities, we obtain the result that is sought, viz:

$$\text{arc : sin}\,(x) \pm \text{arc : sin}\,(x')$$
$$= \text{arc : sin}\left(x\sqrt{(1-x'^2)} \pm x'\sqrt{(1-x^2)}\right) \qquad 1$$
$$\text{arc : cos}\,(y) \pm \text{arc : cos}\,(y')$$
$$= \text{arc : sin}\left(y'\sqrt{(1-y^2)} \pm y\sqrt{(1-y'^2)}\right) \qquad 2$$

From the formula for the cosine, F, No. 2, we obtain by a similar process:

$$\cos(a \pm b) = yy' \mp xx' = yy' \mp \sqrt{(1-y^2)}\sqrt{(1-y'^2)}$$
$$= \sqrt{(1-x^2)}\sqrt{(1-x'^2)} \mp xx'$$

Whence

$$\cos\left(\text{arc : sin}\,(x) \pm \text{arc : sin}\,(x')\right) = \sqrt{(1-x^2)}\sqrt{(1-x'^2)} \mp xx'$$
$$\cos\left(\text{arc : cos}\,(y) \pm \text{arc : cos}\,(y')\right) = yy' \mp \sqrt{(1-y^2)}\sqrt{(1-y'^2)}$$

And as final results:

$$\text{arc : sin}\,(x) \pm \text{arc : sin}\,(x')$$
$$= \text{arc : cos}\left(\sqrt{(1-x^2)}\sqrt{(1-x'^2)} \pm xx'\right) \qquad 3$$
$$\text{arc : cos}\,(y) \pm \text{arc : cos}\,(y')$$
$$= \text{arc : cos}\left(yy' \mp \sqrt{(1-y^2)}\sqrt{(1-y'^2)}\right) \qquad 4$$

The formula F, No. 4, gives:

$$\tan(a \pm b) = \frac{1 \pm z'\dfrac{1}{z}}{\dfrac{1}{z} \mp z'} = \frac{z \pm z'}{1 \mp zz'}$$

(The three next formulæ give forms that are identical.)
We obtain from this last expression:

$$\tan\left(\text{arc : tan}\,(z) \pm \text{arc : tan}\,(z')\right) = \frac{z \pm z'}{1 \mp zz'}$$

And finally:

$$\text{arc : tan}\,(z) \pm \text{arc : tan}\,(z') = \text{arc : tan}\left(\frac{z \pm z'}{1 \mp zz'}\right) \qquad 5$$

It will be readily conceived, that the use of these expres-

sions, combined in every possible manner, and even the introduction of the values of the tangents, instead of the values of the sines and cosines, and conversely, would furnish a multiplicity of formulæ; but they would become either complicated or identical; for it must be considered, that we do not treat of the quantities themselves, but of the form of their combinations.

It will be seen, that in this point of view, the formulæ U, 1 and 2, are already identical; for each of them shows, in each of its terms, a simple quantity, and the square root of Unity diminished by a square.

This is not the case in the formulæ deduced from the cosine; for in them the different products have different signs. The cotangents will evidently give the inverse of the tangents; and the same is the case with the secant and cosecant, in relation to the cosines and the sines.

§ 42. Applying this process to the series Q, we obtain:

By Q, No. 1; $\sin 2a = 2xy = 2x\sqrt{(1-x^2)} = 2y\sqrt{(1-y^2)}$

which gives

$$\sin\big(2 \text{ arc} : \sin(x)\big) = 2x\sqrt{(1-x^2)}$$

whence

6
$$\text{arc} : \sin(x) = \tfrac{1}{2} \text{ arc} : \sin\big(2x\sqrt{(1-x^2)}\big)$$

and

$$\text{arc} : \cos(y) = \tfrac{1}{2} \text{ arc} : \sin\big(2y\sqrt{(1-y^2)}\big)$$

By Q, No. 3 & 4;

$$\cos 2a = 1 - 2x^2 = 1 - 2\sqrt{(1-y^2)}$$
$$= 2\sqrt{(1-x^2)} - 1 = 2y^2 - 1$$
$$\cos\big(2 \text{ arc} : \sin(x)\big) = 1 - 2x^2 = 2\sqrt{(1-x^2)} - 1$$
$$\cos\big(2 \text{ arc} : \cos(y)\big) = 1 - 2\sqrt{(1-y^2)} = 2y^2 - 1$$

and from these we obtain

7
$$\text{arc} : \sin(x) = \tfrac{1}{2} \text{ arc} : \cos(1 - 2x^2)$$
$$= \tfrac{1}{2} \text{ arc} : \cos\big(2\sqrt{(1-x^2)} - 1\big)$$

$$\text{arc} : \cos(y) = \tfrac{1}{2}\,\text{arc} : \cos\left(1 - 2\sqrt{(1-y^2)}\right)$$
$$= \tfrac{1}{2}\,\text{arc} : \cos(2y^2 - 1) \qquad 8$$

By Q, No. 5; $\tan 2a = \dfrac{2}{\dfrac{1}{z} - z} = \dfrac{2z}{1 - z^2}$

whence

$$\tan\left(2\,\text{arc} : \tan(z)\right) = \frac{2z}{1 - z^2}$$

$$\text{arc} : \tan z = \tfrac{1}{2}\,\text{arc} : \tan\left(\frac{2z}{1 - z^2}\right) \qquad 9$$

These formulæ, which may besides be useful, will suffice to give an idea of this application of Trigonometry in the analysis of infinitesimals.

CHAPTER XI.

General Formulæ for the Trigonometric Functions of Multiple Angles, in Terms of the Functions of the Simple Angle only.

§ 43. In chapter VII. we have exhibited general and simple formulæ for the sine, cosine, and tangent, of a multiple angle, in terms of the functions of its aliquot parts; and likewise their application to successive multiples, in the series P.

By successive substitutions and reductions, we might deduce from these, expressions in terms of the functions of the simple angle only.

For the sake of greater simplicity, we shall here deduce these formulæ from those of series F, No. 1 and 2, alone. After a small number of multiples have been examined, a general law will be discovered, by which the determinatior

of the numerical coefficients, and the order in which the powers of the several functions succeed each other, is prescribed. This law will thus be established by induction; and in the succeeding chapter we shall apply it, in order to give an idea of the manner in which trigonometric tables may be constructed; a subject whose principles must be understood, although it cannot be here treated of in all its details.

It may easily be seen, that a variety of formulæ might be obtained, but we shall treat of those only which are the most simple.

§ 44. If, in the formulæ F, No. 1 and 2, we assume $a = b$, we have seen in section 34, by the formulæ Q, No. 1 and 2, that:

$$\sin 2a = 2 \sin a \cos a$$
$$\cos 2a = \cos^2 a - \sin^2 a$$

Continuing this process, by means of successive assumptions, such as

$$b = 2a; \text{ whence } a + b = 3a$$

we obtain for this case in the first place, by F, No. 1:

$$\sin 3a = \sin 2a \cos a + \cos 2a \sin a$$

and

$$\cos 3a = \cos 2a \cos a - \sin 2a \sin a$$

Substituting the values of sine $2a$, and cosine $2a$, in conformity to the value just preceding, we obtain:

$$\sin 3a = 2 \sin a \cos^2 a + \sin a \cos^2 a - \sin^3 a$$
$$= 3 \sin a \cos^2 a - \sin^3 a$$

and

$$\cos 3a = \cos^3 a - \cos a \sin^2 a - 2 \sin^2 a \cos a$$
$$= \cos^3 a - 3 \sin^2 a \cos a$$

For a quadruple angle we shall have:

$$\sin 4a = \sin 3a \cos a + \cos 3a \sin a$$
$$\cos 4a = \cos 3a \cos a - \sin 3a \sin a$$

And substituting from the last:

$$\sin 4a = 3 \sin a \cos^3 a - \sin^3 a \cos a + \cos^3 a \sin a - 3 \sin^3 a \cos a$$

Reducing:

$$\sin 4a = 4 \sin a \cos^3 a - 4 \sin^3 a \cos a$$

in like manner:

$$\cos 4a = \cos^4 a - 3 \sin^2 a \cos^2 a - 3 \cos^2 a \sin^2 a + \sin^4 a$$
$$= \cos^4 a - 6 \sin^2 a \cos^2 a + \sin^4 a$$

We obtain for a quintuple angle the following results in succession:

$$\sin 5a = \sin 4a \cos a + \cos 4a \sin a$$
$$= 4 \sin a \cos^4 a - 4 \sin^3 a \cos^2 a + \sin a \cos^4 a - 6 \sin^3 a \cos^2 a + \sin^4 a$$
$$= 5 \sin a \cos^4 a - 10 \sin^3 a \cos^2 a + \sin^5 a$$

and

$$\cos 5a = \cos 4a \cos a - \sin 4a \sin a$$
$$= \cos^5 a - 6 \sin^2 a \cos^3 a + \sin^4 a \cos a - 4 \sin^2 a \cos^3 a + 4 \sin^4 a \cos a$$
$$= \cos^5 a - 10 \sin^2 a \cos^3 a + 5 \sin^4 a \cos a$$

It is easy to extend this calculation to the subsequent multiples, for which reason we shall only give the results.

$$\sin 6a = 6 \sin a \cos^5 a - 20 \sin^3 a \cos^3 a + 6 \sin^5 a \cos a$$
$$\cos 6a = \cos^6 a - 15 \sin^2 a \cos^4 a + 15 \sin^4 a \cos^2 a - \sin^6 a$$
$$\sin 7a = 7 \sin a \cos^6 a - 35 \sin^3 a \cos^4 a + 21 \sin^5 a \cos^2 a - \sin^7 a$$
$$\cos 7a = \cos^7 a - 21 \cos^5 a \sin^2 a + 35 \cos^3 a \sin^4 a - 7 \cos a \sin^6 a$$

The simplest, and therefore the best, mode of considering the tangent, is, by dividing the sine by the cosine, and reducing. By this process we have:

$$\tan 2a = \frac{\sin 2a}{\cos 2a} = \frac{2 \sin a \cos a}{\cos^2 a - \sin^2 a} = \frac{2 \tan a}{1 - \tan^2 a}$$

$$\tan 3a = \frac{3 \sin a \cos^2 a - \sin^3 a}{\cos^3 a - 3 \sin^2 a \cos a} = \frac{3 \tan a - \tan^3 a}{1 - 3 \tan^2 a}$$

To obtain the second result, we divide constantly by that power of the cosine which constitutes the first term of the denominator.

It may be proper, before proceeding farther, to unite all these results in a table; by which their use may be facilitated, in obtaining conclusions from them for the general formulæ that are the object of this investigation.

$$1 \begin{cases} \sin a = \sin a \\ \sin 2a = 2 \sin a \cos a \\ \sin 3a = 3 \sin a \cos^2 a - \sin^3 a \\ \sin 4a = 4 \sin a \cos^3 a - 4 \sin^3 a \cos a \\ \sin 5a = 5 \sin a \cos^4 a - 10 \sin^3 a \cos^2 a + \sin^5 a \\ \sin 6a = 6 \sin a \cos^5 a - 20 \sin^3 a \cos^3 a + 6 \sin^5 a \cos a \\ \sin 7a = 7 \sin a \cos^6 a - 35 \sin^3 a \cos^4 a + 21 \sin^5 a \cos^2 a - \sin^7 a \\ \sin 8a = \&c. \end{cases}$$

$$2 \begin{cases} \cos a = \cos a \\ \cos 2a = \cos^2 a - \sin^2 a \\ \cos 3a = \cos^3 a - 3 \sin^2 a \cos a \\ \cos 4a = \cos^4 a - 6 \sin^2 a \cos^2 a + \sin^4 a \\ \cos 5a = \cos^5 a - 10 \sin^2 a \cos^3 a + 5 \sin^4 a \cos a \\ \cos 6a = \cos^6 a - 15 \sin^2 a \cos^4 a + 15 \sin^4 a \cos^2 a - \sin^6 a \\ \cos 7a = \cos^7 a - 21 \sin^2 a \cos^5 a + 35 \sin^4 a \cos^3 a - 7 \sin^6 a \cos a \\ \cos 8a = \&c. \end{cases}$$

$$\left.\begin{aligned}
\tan a &= \tan a \\
\tan 2a &= \frac{2\tan a}{1-\tan^2 a} \\
\tan 3a &= \frac{3\tan a-\tan^3 a}{1-3\tan^2 a} \\
\tan 4a &= \frac{4\tan a-4\tan^3 a}{1-6\tan^2 a+\tan^4 a} \\
\tan 5a &= \frac{5\tan a-10\tan^3 a+\tan^5 a}{1-10\tan^2 a+5\tan^4 a} \\
\tan 6a &= \frac{6\tan a-20\tan^3 a+6\tan^5 a}{1-15\tan^2 a+15\tan^4 a-\tan^6 a} \\
\tan 7a &= \frac{7\tan a-35\tan^3 a+21\tan^5 a-\tan^7 a}{1-21\tan^2 a+35\tan^4 a-7\tan^6 a} \\
\tan 8a &= \&c.
\end{aligned}\right\} 3$$

§ 45. If we consider the foregoing formulæ for the sine and cosine of the multiple angles expressed wholly in terms of the sines and cosines of the simple angles, and their successive powers, both in relation to the order in which these powers, and to that in which their coefficients, occur, we shall perceive, that: for every corresponding multiple of the sine and cosine, beginning at the first term of the cosine, thence passing to the first term of the sine, then from the second term of the cosine to the second of the sine, and so on to the end; we have all the terms of the binomial in regular order, as well for the powers of cosine a, and sine a, as for their numeric coefficients; with this difference only, that a regular change of the signs, $+$, and $-$, takes place sepaately, in each of the series.

The same law holds good in the case of the tangents, as far as regards the coefficients; and the powers of the tangents follow in a regular order, from the numerator to the denominator, alternately.

§ 46. We may, therefore, substitute the terms of the binomial theorem in the formulæ, which will express at one view the general law. Calling, then, the number that expresses the multiple of the angle, n, we shall have the following general formulæ, viz:

4 $$\sin na = n \sin a \cos^{n-1} a - \frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3} \sin^3 a \cos^{n-3} a$$

$$+ \frac{n\,(n-1)\,(n-2)\,(n-3)\,(n-4)}{1\,.\,2\,.\,3\,.\,4\,.\,5} \sin^5 a \cos^{n-5} a$$

$$- \frac{n\,(n-1) \ldots\ldots (n-6)}{1\,.\,2 \ldots\ldots\ldots 7} \sin^7 a \cos^{n-7} a$$

5 $$\cos n a = \cos^n a - \frac{n\,(n-1)}{1\,.\,2} \cos^{n-2} a \sin^2 a$$

$$+ \frac{n\,(n-1)\,(n-2)\,(n-3)}{1\,.\,2\,.\,3\,.\,4} \cos^{n-4} a \sin^4 a$$

$$- \frac{n\,(n-1) \ldots\ldots (n-5)}{1\,.\,2 \ldots\ldots\ldots 6} \cos^{n-6} a \sin^6 a$$

$$+ \frac{n\,(n-1) \ldots\ldots (n-7)}{1\,.\,2 \ldots\ldots\ldots 8} \cos^{n-8} a \sin^8 a - \&c. + \&c.$$

Making, in these two series, cosine^n a, a common factor to the whole series, they present series with the powers of the tangent of the simple arc in regular succession; thus:

6 $$\sin n a = \cos^n a \left(n \tan a - \frac{n\,(n-1)\,(n-2)}{2\,.\,3} \tan^3 a \right.$$

$$+ \frac{n\,(n-1)\,(n-2)\,(n-3)\,(n-4)}{2\,.\,3\,.\,4\,.\,5} \tan^5 a$$

$$- \frac{n\,(n-1)\,(n-2) \ldots\ldots (n-6)}{2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7} \tan^7 a$$

$$\left. + \frac{n\,(n-1)\,(n-2) \ldots\ldots (n-8)}{2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7\,.\,8\,.\,9} \tan^9 a - \&c. + \&c. \right)$$

$$\cos n a = \cos^n a \left(1 - \frac{n(n-1)}{2} \tan^2 a \right. \qquad 7$$

$$+ \frac{n(n-1)(n-2)(n-3)}{2 \,.\, 3 \,.\, 4} \tan^4 a$$

$$- \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6} \tan^6 a$$

$$+ \frac{n(n-1)(n-2)(n-3)\ldots\ldots(n-7)}{2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6 \,.\, 7 \,.\, 8} \tan^8 a$$

$$\left. - \frac{n(n-1)\ldots\ldots(n-9)}{2\ldots\ldots\ldots10} \tan^{10} a + \&c. - \&c. \right)$$

By the division of 6 by 7, the series for the tangent becomes:

$$\tan na = \frac{n \tan a - \frac{n(n-1)(n-2)}{2 \,.\, 3} \tan^3 a + \frac{n\ldots\ldots(n-4)}{1\ldots\ldots5} \tan^5 a}{1 - \frac{n(n-1)}{2} \tan^2 a + \frac{n\ldots\ldots(n-3)}{1\ldots\ldots4} \tan^4 a - \frac{n\ldots\ldots(n-5)}{1\ldots\ldots6} \tan^6 a} \qquad 8$$

$$\frac{- \frac{n(n-1)\ldots\ldots(n-6)}{2\ldots\ldots\ldots7} \tan^7 a + \&c.}{+ \frac{n(n-1)\ldots\ldots(n-7)}{2\ldots\ldots\ldots8} \tan^8 a - \&c.}$$

Performing the division which is here indicated, (which may be done most easily by the method of indeterminate coefficients, that will be explained hereafter,) it is immediately discovered, that every subsequent term of the resulting series depends on all the preceding ones, by a combination of the binomial coefficients; the law of which is easy and simple, although the series itself, when developed, becomes long and complicated, though regular. It shall be represented here in the first shape. For this purpose, let the series be represented by the following, with undetermined coefficients, which, it

will be seen, may all be determined from each other when the first is determinate, as is the case.

$$\begin{aligned}\tan n\,a &= A \tan a + B \tan^3 a + C \tan^5 a + D \tan^7 a \\ &\quad + E \tan^9 a + F \tan^{11} a\end{aligned}$$

The determination of these coefficients gives $A = n$; and therefore, for the convenience of writing the result, denote the successive binomial coefficients thus; call:

$$\frac{n-1}{2} = n_1 \;;\; \frac{n-2}{3} = n_2 \;;\; \frac{n-3}{4} = n_3 \;;$$
$$\frac{n-4}{5} = n_4 \;;\; \frac{n-5}{6} = n_5 \;;\; \text{\&c.}$$

Then will the series be represented in the following form:

$$\left\{\begin{aligned} \tan n\,a &= n \tan a + (n\,n_1\,A - n\,n_1\,n_2) \tan^3 a \\ &+ (n\,n_1\,B - n\,n_1\,n_2\,n_3\,A + n\,n_1\,n_2\,n_3\,n_4) \tan^5 a \\ &+ (n\,n_1\,C - n\,n_1\,n_2\,n_3\,B + n\,n_1 \ldots\ldots n_5\,A - n\,n_1 \ldots\ldots n_6) \tan^7 a \\ &+ (n\,n_1\,D - n\,n_1\,n_2\,n_3\,C + n \ldots\ldots n_5\,B - n \ldots\ldots n_7\,A \\ &+ n \ldots\ldots n_8) \tan^9 a + (n\,n_1\,E - n\,n_1\,n_2\,n_3\,D \\ &+ n \ldots\ldots n_5\,C - n \ldots\ldots n_7\,B + n \ldots\ldots n_9\,A - n \ldots\ldots n_8) \tan^{11} a + \text{\&c.} \end{aligned}\right. \tag{9}$$

which may evidently be continued with ease, as the law is apparent. The developement of the factor remaining somewhat complicated, it may be omitted here, particularly as in all calculations of series, the terms are calculated in succession, the first term being always made the largest.

CHAPTER XII.

Elementary Ideas relating to the Construction of Trigonometric Tables.

§ 47. We have already seen, that the circumference of the circle and the trigonometric functions are incommensurable; the latter cannot, therefore, be expressed in parts of the first, and conversely, except by approximation, or by the transcendental or infinitessimal analysis. The first of these methods was employed at first, before the latter had cleared an easy way to results of this nature.

In order, then, to give an idea of the methods that may be employed to determine the trigonometric functions, by the methods of the infinitesimal calculus, it is necessary that we previously give an idea of the form of this calculus.

In the series of formulæ E, we have already seen the expression $\frac{1}{0}$ = Infinity. By it we are to understand, that the quantity it expresses is greater than can be expressed by numbers, in the same way that 0 represents the absence of all quantity. An attentive observation of the values given by the series E, shows the complete circle of all possible quantities; for in it the ratios between the several lines are seen to increase from 0 to ∞, both positive and negative; and undergoing changes of sign in both transitions, through these extreme values. This shows, at the same time, that all consideration of quantity is wholly relative, for it is from a ratio that this result is obtained; a result, as simple as it is valuable, in mathematical researches.

We ground upon the foregoing, a principle of constant use in the infinitesimal calculus; it consists in repelling or rejecting in this calculus every quantity which is not multiplied by ∞, as not appertaining to the hypothesis on which this calculus is founded.

Admitting then ∞ into the series of symbols, that express quantity, we shall have $\frac{1}{\infty}$ equal to what is called: infinitely small, and as in every species of calculus we must employ the conventional signs in conformity with their conventional signification, and the value attributed to them, we have

$$\infty \frac{1}{\infty} = 1 \text{ ; or } \frac{a\,\infty}{\infty} = a \text{ ;}$$

that is to say, a quantity divided by ∞, and multiplied by ∞, is equal to the quantity itself, as is the case with any other number; all this is no more than a form of calculus, by which the properties of ∞ are determined, or agreed upon, as is the case with all other symbols that represent quantity. We must never lose sight of the principle, that in analysis in general we only consider the form of the combinations of quantities, without regard to the quantities themselves; except so far as regards their ratios to certain other quantities, that are compared or placed in relation with them.

When what has been said above, is once understood, it will be easy to comprehend the use that is made of these principles, in this chapter; in which we show the manner of determining, by means of the general formulæ already given, the trignometric functions of a given angle or arc, upon the assumption, that the value of π is known; which supposition is then justified by the inversion of these results, so as to determine the value of π, by means of the preceding formulæ, expressed in parts of the radius taken equal to Unity.

§ 48. In conformity with the principles just laid down, it will be sufficient to assume, in the series W, No. 4, 5, 6, and 7, the arc $a = \frac{1}{\infty}$; that is to say, infinitely small; and the number by which it is multiplied, $n = \infty$, and investigate the consequences of this hypothesis, in conformity with the principles and formulæ already explained.

It will be ascertained, by means of the values found in the series E, that when $a = \frac{1}{\infty}$; the value of $\cos a = 1$; the sine or the tangent is equal to the arc itself, being all perpendiculars at the end of the radius $= 1$, and indefinitely small.

This supposition will transform the formulæ quoted, into such as contain only the powers of the quantity, $(n\,a,)$ and the numbers that are found in the denominators of the several terms. (To make this more clear, as well as for the sake of brevity, we shall here suppose $n\,a = x$.)

By this method of proceeding, the formula W, No. 4, is transformed thus:

X

$$\sin x = x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \frac{x^7}{2.3.4.5.6.7} + \frac{x^9}{2.4.5.6.7.8.9} - \frac{x^{11}}{2.3\ldots\ldots10.11} + \&c. \qquad 1$$

and the formula W, No. 5, into the following:

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \frac{x^6}{2.3.4.5.6} + \frac{x^8}{2.3.4.5.6.7.8} - \frac{x^{10}}{2.3\ldots\ldots9.10} + \frac{x^{12}}{2.3\ldots\ldots11.12} + \&c. \qquad 2$$

The formula for the tangent W, No. 9, becomes

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{3.5} + \frac{17\,x^7}{3^2.5.7} + \frac{2.31\,x^9}{3^2.5.7.9} + \frac{2.691.x^{11}}{3^2.5^2.7.9.11} + \frac{2.8447.x^{13}}{3^3.5^2.7.9.11.13} + \&c. \qquad 3$$

§ 49. These series then give, by approximation, the sine, cosine, and tangent of an arc $(n\,a = x)$ which is supposed to

be given under the hypothesis, that $n = \infty$, and $a = \frac{1}{\infty}$; this arc being expressed in the same terms as π, or in terms of the radius.

The inspection of the formulæ that present increasing powers of this arc, shows, that in order to obtain a series whose following terms shall each be less than that which precedes it, $(x,)$ must be a fraction, the powers of which constantly decrease; this will always be the case here, as the arc equal to the radius, which is the unity in which π is given, is more than $57°.17'.44''$, 8, and we have seen, (in chapter 9,) that the trigonometric functions need at farthest be calculated to 45°. A multiplicity of formulæ give the values of the trigonometric functions of compound and multiple arcs, from the functions of their parts; it is therefore sufficient to calculate the latter, by those series, in a proper manner, to obtain all those which may be necessary.

The smaller the arc x, the fewer terms of the series will be needed, in order to obtain the value accurately to a given number of decimal places: but in this respect it is obviously necessary: that, as the result of the first calculation is to be multiplied, to obtain functions of the greater arcs, we must give to the first calculation or value, a proportionally greater number of decimals.

These explanations will suffice to give an idea of the manner in which trigonometric tables may be calculated, which is all that we need illustrate in this treatise.

§ 50. The last condition that remains, is, to determine the value of π, in terms of the diameter, or rather, of $\frac{1}{2}\pi$, in terms of the radius, which is evidently the same thing. The series which have been pointed out, perform this office also, by means of the process in calculation, called the inversion of series. This process has in itself no difficulty; it will be explained by the application which shall here be made of it, in relation to the last of the above series, which

is chosen here, on account of its leading by the most converging series, to the end here proposed.

To do this, a series is supposed given in the form in which it may be always easily foreseen that it will assume, in which the coefficients are indeterminate, and become determined in the course of the process.

For this purpose we shall assume, in the present case,

$$x = A \tan x + B \tan^3 x + C \tan^5 x + D \tan^7 x + E \tan^9 x + \&c.$$

We shall next express the several values of tangent x, tangent ^3x, &c. by the preceding series, X, No. 3, and its corresponding powers, using no more terms than are necessary to determine the law of the progression of the coefficients. The sum of all the terms, will evidently give a new value of x, and subtracting x from both sides of the equation, the value of the resulting series becomes $= 0$; from this it necessarily follows, that each sum of the terms of the various powers of tangent x will itself be $= 0$, since the equation must be true whatever be the value of tangent x. From this consequence are deduced as many equations as there are undetermined coefficients; these coefficients therefore become determinable in succession; the value of the entire series will, in consequence, be determined by the insertion of those values in their proper places.

The following is the process:

Expressing the several values of the powers of tangent x, by the multiplication of series No. 3, itself, we obtain, after multiplying each by its appropriate coefficient,

$$A \tan x = Ax + \frac{A}{3}x^3 + \frac{2A}{3.5}x^5 + \frac{17A}{3^2.5.7}x^7 + \frac{62A}{3^2.5.7.9}x^9 +$$

$$B \tan^3 x = Bx^3 + Bx^5 + \frac{11B}{3.5}x^7 + \frac{88B}{3^3.7}x^9 +$$

$$C \tan^5 x = + Cx^5 + \frac{5C}{3}x^7 + \frac{16C}{3^2}x^9 +$$

$$D \tan^7 x = \qquad D\, x^7 + \frac{7D}{3} x^9 +$$

$$E \tan^9 x = \qquad + E\, x^9 +$$

$$F \tan^{11} x = \qquad +$$

&c. &c.

The series given above for x, becomes thus equal to the sum of all these terms, and subtracting x from both sides of the equation

$$0 = (A - 1)\, x + \left(\frac{A}{3} + B\right) x^3 + \left(\frac{2\,A}{3.5} + B + C\right) x^5$$
$$+ \left(\frac{17\,A}{3^2.5.7} + \frac{11\,B}{3.5} + \frac{5\,C}{3} + D\right) x^7$$
$$+ \left(\frac{62\,A}{3^2.5.7.9} + \frac{88\,B}{3^3.7} + \frac{16\,C}{3^2} + \frac{7\,D}{3} + E\right) x^9 + \&c.$$

We thus obtain, for the determination of these coefficients, the following successive equations and results :

$$A - 1 = 0 \qquad \text{whence} \qquad A = 1$$

$$\frac{A}{3} + B = 0 \qquad B = -\frac{1}{3}$$

$$\frac{2\,A}{3.5} + B + C = 0 \qquad C = +\frac{1}{5}$$

$$\frac{17\,A}{3^2.5.7} + \frac{11\,B}{3.5} + \frac{5\,C}{3} + D = 0 \qquad D = -\frac{1}{7}$$

$$\frac{62\,A}{3^2.5.7.9} + \frac{88\,B}{3^3.7} + \frac{16\,C}{3^2} + \frac{7\,D}{3} + E = 0 \qquad E = +\frac{1}{9}$$

&c. &c.

Substituting these values of the coefficients, in the series assumed for the value of x, we finally obtain

$$x = \tan x - \tfrac{1}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x - \tfrac{1}{7}\tan^7 x + \tfrac{1}{9}\tan^9 x - \tfrac{1}{11}\tan^{11} x + \&c. \qquad 4$$

§ 51. To make use of this series for the purpose of determining the value of π, we may make use of several methods, drawn from the combination of different arcs, whose tangents are rational, or expressed in simple fractions. As an example we shall only choose the following:

Assuming the arc of 45°, whose tangent = 1, to be compounded of two other arcs, the tangent of one of which $\tan a = \frac{1}{2}$; we have by formula F, No. 6,

$$\tan(a + b) = 1 = \frac{\frac{1}{2} + \tan b}{1 - \frac{1}{2}\tan b}$$

which gives the value of

$$\tan b = \tfrac{1}{3}$$

We therefore have two tangents, expressed in simple fractions, that may be introduced into the series X, No. 4, and whose sum will give the value of the arc of 45°. Four times this value is the half circumference, or the value of π, which is the quantity sought, expressed in terms of the radius = 1. This substitution in the series gives

$$\pi = 4\left\{\begin{array}{l} \frac{1}{2} - \frac{1}{3}(\frac{1}{2})^3 + \frac{1}{5}(\frac{1}{2})^5 - \frac{1}{7}(\frac{1}{2})^7 + \frac{1}{9}(\frac{1}{2})^9 - \frac{1}{11}(\frac{1}{2})^{11} \\ \frac{1}{3} - \frac{1}{3}(\frac{1}{3})^3 + \frac{1}{5}(\frac{1}{3})^5 - \frac{1}{7}(\frac{1}{3})^7 + \frac{1}{9}(\frac{1}{3})^9 - \frac{1}{11}(\frac{1}{3})^{11} \\ \quad + \frac{1}{13}(\frac{1}{2})^{13} - \frac{1}{15}(\frac{1}{2})^{15} + \&c. \\ \quad + \frac{1}{13}(\frac{1}{3})^{13} - \frac{1}{15}(\frac{1}{3})^{15} + \&c. \end{array}\right\} \qquad 5$$

Performing these calculations to a sufficient number of terms, we obtain, in whole numbers and decimals,

$$\pi = 3{,}1415926535897932384626433832795, \&c. \qquad 6$$

This being the value of an arc of 180°, in terms of the radius, it may be seen how the value of any arc whatsoever, may be expressed in parts of the same radius, by taking a proportional part of this value of 180°. This value of an

arc $= x$, may then be inserted in the series X, No. 1, 2, 3, to calculate from it the *sine*, *cosine*, or *tangent*, in conformity to the assumptions that have been made in speaking of these series.

§ 52. We have thus completed the circle of our analytic investigation, in relation to the trigonometric functions; beginning with the elementary definitions, or functions, deduced from the ratios existing between the sides of a right-angled plane triangle, and proceeding until we reach the determination of the value of the circumference of the circle in terms of the radius; which last value serves as a foundation for the calculation of trigonometric functions in actual numbers.

By means of series more advantageous than those which have been given, but whose investigation here would lead us too far, the number π, has been calculated to 148. places of decimals; an accuracy far beyond what is ever necessary in any calculation whatever; and which, even more than ever, renders it useless to search for the quadrature of the circle.

It cannot belong to this treatise to treat of logarithmic series for the trigonometric functions; logarithms in general, forming no part of our plan, would introduce a complication that is not intended. In the 4th part of this treatise, their knowledge is, however, necessarily supposed, at least so far as all trigonometric tables usually teach, in their introduction.

PART II.

PLANE TRIGONOMETRY.

CHAPTER I.

Solution of all the Cases of Oblique Angled Plane Trigonometry.

§ 53. PROVIDED with the analysis, the results, and formulæ of the foregoing chapters, Oblique Angled Plane Trigonometry becomes an easy application of the formulæ we have obtained to the solution of all its several problems. We shall here treat of it in this point of view.

In order to determine a lineal dimension, it is necessary that one of the given quantities should be also a lineal dimension.

To determine the absolute value of a triangle, we must therefore have, among the data, one of its sides, for it is well known, by the elementary principles of geometry, that the determination of the angles only, determines nothing more than the similarity of the triangles.

We now give the various problems with their solutions.

§ 54. *Problem* 1. To find the relation between the sides and one of the trignometric functions of the angles of an oblique angled plane triangle.

Let ABC, (figure 6, and 7,) be an oblique angled plane triangle, whose sides are a, b, c, respectively opposite the angles of the same name. From the point A let fall the perpendicular $AD = d$ upon the opposite side $BC = a$, or upon that side produced, if the angle B, or C, be obtuse, (as in fig.

7,) the triangle ABC will furnish two right angled triangles, ABD, and ACD, having the side, $AD = d$, common.

Solution. By the formula No. 1, of the series A, or first definition, we have in the two triangles, and in both cases, (since the sine of an angle is equal to the sine of its supplement.)

$$\frac{d}{b} = \sin C \; ; \qquad \text{and} \qquad \frac{d}{c} = \sin B$$

Therefore :

$$d = b \, . \sin C = c \, . \sin B$$

Or, expressed in a proportion :

Y

$$b : c = \sin B : \sin C$$

And this being general for any side, we have also :

1

$$b : a = \sin B : \sin A$$

$$a : c = \sin A : \sin C$$

This is generally expressed thus :

In any plane triangle, the sides are to each other as the sines of their opposite angles. It gives therefore the solution of all the cases, where two of the parts given are opposite to each other, and the part to be found opposite to its corresponding given part.

Corollary. If from the three angular points of a triangle, perpendiculars be let fall upon the opposite sides, these perpendiculars are to each other in the inverse ratio of the sides on which they fall.

In the triangle ABC, (fig. 8,) let d, d', be the perpendiculars falling upon the sides a, and b, respectively ; we have by A, No. 1, as before,

$$\sin C = \frac{d}{b} = \frac{d'}{a} \; ; \quad \text{or} \quad d : d' = b : a$$

§ 55. *Problem* 2. In an oblique angled plane triangle, given

the two sides and the included angle, to find the two remaining angles.

The sum of the three angles of a plane triangle is always equal to two right angles; (elementary geometry;) in this case then we have given not only the two sides, and the angle included, but also the sum of the two angles sought; all then that is necessary to determine each angle separately, is to find their difference; for the largest is equal to half the sum increased by half the difference, and the least to half the sum diminished by half the difference.

In the same triangle that has been used in the first problem, having given A, b, and c, we have, as has been demonstrated,

$$b : c = \sin B : \sin C$$

And by composition of this proportion,

$$b + c : b - c = \sin B + \sin C : \sin B - \sin C$$

Substituting from series N, No. 3,

$$b + c : b - c = \tan \tfrac{1}{2} (B + C) : \tan \tfrac{1}{2} (B - C)$$

whence:

$$\tan \tfrac{1}{2} (B - C) = \tan \tfrac{1}{2} (B + C) \frac{b - c}{b + c}$$

And since the three angles, $A + B + C = 180°$,

or $\tfrac{1}{2} A + \tfrac{1}{2} (B + C) = 90°$,

we have: $90° - \tfrac{1}{2} A = \tfrac{1}{2} (B + C)$

and $\tan \tfrac{1}{2} (B + C) = \tan (90° - \tfrac{1}{2} A) = \cot \tfrac{1}{2} A$

The formula becomes:

$$\tan \tfrac{1}{2} (B - C) = \cot \tfrac{1}{2} A \frac{b - c}{b + c} \qquad 2$$

And calling $\frac{d}{2} = \frac{B - C}{2}$; we have the two angles, avoid-

ing all unnecessary subtraction, (and B being considered the greater angle.)

$$B = 90^\circ - \tfrac{1}{2} A + \tfrac{1}{2} d$$
$$C = 90^\circ - \tfrac{1}{2} A - \tfrac{1}{2} d$$

This formula requires in its use, an addition and a subtraction. It may, when desired, be adapted to the calculation of quantities given in logarithms, a case that occurs in astronomy, by the following transformations.

For this purpose, we divide the numerator and denominator of the fractional part of the formula by b; and we have:

$$\tan \tfrac{1}{2}(B - C) = \cot \tfrac{1}{2} A \, \frac{1 - \dfrac{c}{b}}{1 + \dfrac{c}{b}}$$

Comparing this form of the fractional part with the formula S, No. 18, using the lower signs, it will be found: that,

3 if we call $\dfrac{c}{b}$ = tangent z, we have:

$$\tan (45^\circ - z) = \frac{1 - \tan z}{1 + \tan z}$$

Taking, then, for the calculation of the data given in logarithms, tangent $z = \dfrac{c}{b}$, we obtain for the solution of this case the formula:

4 $$\tan \tfrac{1}{2}(B - C) = \cot \tfrac{1}{2} A \tan (45^\circ - z)$$

This is an instance of the application of the preceding analytic formulæ for trigonometric functions, to the transformation of an expression containing addition and subtraction into one that can be calculated by logarithms alone; and

we shall always have $\frac{c}{b} \angle 1$, according to the original supposition of $B \geq C$.

§ 56. *Problem* 3. Given two sides and the included angle, to find the third side.

Given, a, c, and B, to find b; in the same triangle, let d denote the perpendicular upon a, as in the first problem.

By the elementary formulæ of series A, we have:

$$\frac{BD}{c} = \cos B \; ; \quad \text{thence} \quad BD = c.\cos B$$

and
$$\frac{d}{c} = \sin B \; ; \qquad d = c.\sin B$$

and
$$CD = a - BD = a - c.\cos B$$

By Geometry:

$$\begin{aligned} b^2 &= d^2 + CD^2 = c^2.\sin^2 B + (a - c.\cos B)^2 \\ &= c^2 \sin^2 B + a^2 + c^2 \cos^2 B - 2a.c.\cos B \\ &= c^2 (\sin^2 B + \cos^2 B) + a^2 - 2a.c \cos B \end{aligned}$$

And because *sine*^{2} + *cosine*^{2} = 1:

$$b^2 = c^2 + a^2 - 2a.c.\cos B \qquad 5$$

This formula, which would be very inconvenient to calculate, may be reduced to an easy form for logarithmic calculation, in two different ways. These will be obvious, when we consider that the formulæ Q, No. 4 and 3, give two different values for cosine B, the one in the sine, the other in the cosine, of the half angle.

Taking, then, in the first place:

$$\cos B = 1 - 2 \sin^2 \tfrac{1}{2} B$$

And substituting this value in the equation for cosine B, we have:

$$b^2 = c^2 + a^2 - 2ac + 4.c.a.\sin^2 \tfrac{1}{2} B$$
$$= (a \backsim c)^2 + 4.a.c.\sin^2 \tfrac{1}{2} B$$

And using $(a \backsim c)$ as a factor common to both the right hand terms:

$$b^2 = (a \backsim c)^2 \left(1 + \frac{4.c.a.\sin^2 \frac{1}{2} B}{(a \backsim c)}\right)$$

Extracting the root:

6 $$b = (a \backsim c) \left(1 + \frac{4.c.a.\sin^2 \frac{1}{2} B}{(a \backsim c)^2}\right)^{\frac{1}{2}}$$

It may be seen: that, by the method used in the preceding Problem, this formula may be adapted to the use of logarithms, employing the consideration: that, according to C, No. 8, we have:

$$\sec^2 = 1 + \tan^2 = \frac{1}{\cos^2}$$

If, then, we make

$$\tan^2 x = \frac{4.a.c.\sin^2 \frac{1}{2} B}{(a \backsim c)^2}$$

7 or $$\tan x = \frac{2 \sin \frac{1}{2} B}{a \backsim c} \sqrt{(a.c)}$$

the final formula will become:

8 $$b = \frac{a \backsim c}{\cos x}$$

Second Transformation. Taking:

$$\cos B = 2 \cos^2 \tfrac{1}{2} B - 1$$

and substituting this value in No. 5, the formula becomes:

$$b^2 = a^2 + c^2 + 2ac - 4.a.c.\cos^2 \tfrac{1}{2} B$$

And by processes exactly similar to those used for the formula 6, it will finally become:

$$b = (a + c)\left(1 - \frac{4.c.a.\cos\frac{1}{2}B}{(a+c)^2}\right)^{\frac{1}{2}} \qquad 9$$

Here we have, by the same means as in the foregoing transformation, representing the second term under the radical as the square of a sine or cosine, according to C, No. 4 or 5, and by an analogous process:

$$\cos^2 x = \frac{4.c.a.\cos^2\frac{1}{2}B}{(a+c)^2}$$

or

$$\cos x = \frac{2\cos\frac{1}{2}B}{a+c}(a.c)^{\frac{1}{2}} \qquad 10$$

And finally:

$$b = (c + a)\sin x \qquad 11$$

If the distances, a and c, were given in logarithms, (as may occur in astronomy,) it will be observed, that in applying to the part $(a \pm c)$, in the two formulæ 8 and 11, the same mode of transformation for the application of an auxiliary angle, by writing $(a \pm c) = a\left(1 \pm \frac{c}{a}\right)$ in both the places where it occurs, these formulæ will be transformed, and fitted for that use, upon the same principles, every thing else remaining as above.

§ 57. *Problem* 4. Given, the three sides of a plane triangle, to determine one of the angles, (suppose the angle B.)

Solution. The formula No. 5 gives this solution by simple transposition; for, from

$$b^2 = a^2 + c^2 - 2ac\cos B$$

it follows, that

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} \qquad 12$$

This formula would also be inconvenient to calculate; it has the same two modes of transformation as the preceding, by

the insertion of the sine or cosine of the half angle; and a third is obtained by the division of these two.

First Transformation. From the preparation for formula 6, is first obtained by transposition and division:

$$\sin^2 \tfrac{1}{2} B = \frac{b^2 - (a \backsim c)^2}{4.a.c}$$

The two terms of the numerator being the difference of two squares, the product of the sum and difference of their roots can be substituted for them; then, dividing the numerical coefficient of the denominator into two factors, and applying them to the two factors of the numerator, we obtain:

13 $$\sin^2 \tfrac{1}{2} B = \frac{\left(\frac{a + b - c}{2}\right)\left(\frac{b + c - a}{2}\right)}{a.c}$$

If we assume $p = \dfrac{a + b + c}{2}$, the expression becomes still simpler, for this assumption gives:

14 $$p - c = \frac{a + b - c}{2}; \text{ and } p - a = \frac{b + c - a}{2}.$$

Inserting this in the formula, it becomes:

$$\sin^2 \tfrac{1}{2} B = \frac{(p - a)(p - c)}{a.c}$$

And extracting the root:

15 $$\sin \tfrac{1}{2} B = \left(\frac{(p - a)(p - c)}{a.c}\right)^{\frac{1}{2}}$$

Second Transformation. From the preparation for formula 9, we obtain, by transposition and division:

$$\cos^2 \tfrac{1}{2} B = \frac{(a + c)^2 - b^2}{4.a.c}$$

And by steps exactly analogous to the foregoing, and the supposition that $p = \dfrac{a+b+c}{2}$, as above, we obtain here:

$$\cos^2 \tfrac{1}{2} B = \frac{\left(\dfrac{a+c+b}{2}\right)\left(\dfrac{a+c-b}{2}\right)}{ac}$$

$$= \frac{p\,(p-b)}{ac}$$

$$\cos \tfrac{1}{2} B = \left(\frac{p\,(p-b)}{ac}\right)^{\frac{1}{2}} \qquad 16$$

When these formulæ are employed, we must choose that which gives the greatest degree of exactness in the given case. The principle which will determine this choice, is as follows: when the angle is small, the sine varies rapidly, while the cosine varies but little. In this case we should use the formula that employs sine $\frac{1}{2} B$; if the angle that is sought is obtuse, we must use the formula that gives the value of cosine $\frac{1}{2} B$, because it is the cosine that varies most in this case.

It will be observed, that, in general, the formulæ that determine the half angles have the advantage of avoiding all ambiguity between obtuse or acute angles; because they can never be obtuse, as in that case $B \geq 180°$, which is impossible.

We may deduce from these two formulæ a third, giving the tangent $\frac{1}{2} B$, that is applicable with equal advantage to all cases. For it is evident, that:

$$\frac{\sin \frac{1}{2} B}{\cos \frac{1}{2} B} = \tan \tfrac{1}{2} B = \left(\frac{(p-a)\,(p-c)}{p\,(p-b)}\right)^{\frac{1}{2}} \qquad 17$$

This requires two more preliminary steps to prepare the data for calculation; but they are short.

§ 58. *Problem* 5. Given, two sides, and an angle opposite to one of the sides; to find the third side.

C, *c*, and *b*, being given, to find *a*.

We find already, from inspection of *Fig.* 6 *and* 7, that the result may, in this case, be doubtful; as with these data, the angle at *B* may be either acute or obtuse, with *c* of equal magnitude, since this line is always greater than the perpendicular *d*, and, being drawn from the same point *A*, may cut *CB* on either side of this perpendicular; which will give two values of *CB*, that are equally possible, and both given by the formula. Circumstances, other than those of the mere magnitude of the parts given, will determine which of the two results is the true one.

Solution. In the formula Y, No. 4, *Problem* 3, we might suppose *b* given, and *a* to be determined, the other data remaining the same, and obtain this solution; this would lead to an equation of the second order. By the following process we reach the same result more easily.

By Y, No. 1, *problem* 1, we have:

$$c : b = \sin C : \sin B$$

whence $$\sin B = \frac{b \sin C}{c}$$

From series A, No. 2, we have, by application of this case:

$$CD = b \cos C$$

and $$BD = c \cos B$$

And according as *B* is acute or obtuse, we find:

$$BC = CD \pm BD = a$$

Expressing this in terms of the above two values, we have:

18 $$a = b \cos C \pm c \cos B$$

And substituting for cosine *B* its value deduced from the first enunciation above, expressing it, in conformity with series C, No. 5, the final result is:

19 $$a = b \cos C \pm c \left(1 - \frac{b^2 \sin^2 C}{c^2}\right)^{\frac{1}{2}}$$

A formula troublesome to calculate, though it may be reduced to logarithmic calculation, by the use of two auxiliary arcs: for we may take:

$$\sin x = \sin B = \frac{b \sin C}{c}$$

and thereby obtain:

$$a = b \cos C \pm c \cos B$$
$$= b \cos C \left(1 \pm \frac{c \cos B}{b \cos C}\right)$$

where we are evidently again to assume, according to the case:

$$\left.\begin{matrix}\tan \\ \text{or} \quad \sin\end{matrix}\right\} y = \left(\frac{c \cos B}{b \cos C}\right)^{\frac{1}{2}} \qquad 20$$

and would ultimately obtain:

$$a = \frac{b \cos C}{\cos^2 y} \qquad 21$$

$$\text{or} \qquad a = b \cos C \cos^2 y$$

But such a calculation would evidently be tedious, and it is far preferable to calculate sine B by *problem* 1, and then a by the two parts of the formula No. 18, above, unless it happen, that only the logarithms of b and c be given; when this mutation will be necessary and applicable.

§ 59. We have thus obtained the solutions of every possible case of Oblique Angled Plane Trigonometry, for data directly given, by means of formulæ affording the greatest facilities for calculation. It will be readily conceived: that what is said of an angle B, or of any side b, &c. is always to be understood, in a general sense, of every other angle or side. It is proper here, again to direct the attention to one general character of all the formulæ, namely: that they always present the linear dimensions in an even number of fac-

ters, and the trigonometric functions in an odd number of factors, when the part to be determined is a trigonometric function; and, conversely, the trigonometric functions in an even number of factors, and the linear dimensions in an odd number of factors, when a linear dimension is to be determined. This is the general and well known character of all proportions; from which, also, analytical formulæ cannot deviate, whatever be the complication of the data contained in them. The trigonometric functions being mere ratios, (that is, generally speaking, numbers,) as has been observed in section 11, they here, form as such, objects of a determined nature.

CHAPTER II.

Calculation of the Surfaces of Plane Triangles from different Data.

§ 60. We think proper to add to the solutions of Trigonometry that give the unknown parts of triangles from those which are known, the solution of the problem: to find the surface of the triangle in every case of the before-mentioned data. This is evidently possible; for if we have the data necessary to determine the unknown parts of a triangle, the same data must also give its contents or surface. We shall take the cases in the same order as in the preceding solutions. It must, in the first place, be recollected, that the surface of the triangle is half the product of one of the sides, assumed as a base, into the perpendicular let fall upon it from the opposite angular point, as is taught in the elements of Geometry. It is, therefore, the principal object of these problems, to express the value of this perpendicular in terms of the parts given. Its product into the given base, divided by two, will, then, always give the surface, or contents of the triangle.

§ 61. *Problem* 1. **Given, two angles and the included side; to find the Surface of the triangle.**

In the triangle ABC, as before, let BC and a be given, to find the surface.

We have by Y, No. 1:

$$b = \frac{a \sin B}{\sin A} = \frac{a \sin B}{\sin (B + C)}$$

For the sine of an angle is equal to the sine of its supplement, not only in magnitude, but also in sign; and because the sum of all the three angles of a plane triangle is equal to two right angles, the sine of any one of the angles is equal to the sine of the sum of the two others.

Calling d the perpendicular let fall upon the side a, we have, as the value of this perpendicular:

$$d = b \sin C = \frac{a \sin B \sin C}{\sin (B + C)}$$

Multiplying this by half the base on which the perpendicular d falls, that is, $\frac{a}{2}$, and calling the surface of the triangle $= S$, we have:

$$S = \frac{a^2 \sin B \sin C}{2 \sin (B + C)} \qquad \text{Z} \; 1$$

And supposing the value of sine $(B + C)$ to be expressed in conformity to F, No. 1, and dividing both numerator and denominator by *sine* B *sine* C, we have also:

$$S = \frac{a^2}{2 (\cot B + \cot C)} \qquad 2$$

But the formula No. 1, is more convenient for logarithmic calculation.

§ 62. *Problem* 2. **Given, two sides and the included angle; to find the Surface of the triangle.**

Given, a, b, and C.

We have by the elementary definitions of series A, the value of the perpendicular:

$$d = b \,.\, \sin C$$

Multiplying this perpendicular by $\dfrac{a}{2}$, we have for the surface:

3 $$S = \frac{ab \sin C}{2}$$

§ 63. *Problem* 3. Given, the three sides; to find the Surface of the triangle.

By the theorem of geometry, so often employed: that the square of the hypothenuse is equal to the sum of the squares of the two sides, and expressing the parts by trigonometry, as in section 56, we find:

$$BD = c \,.\, \cos B$$

We have thence:

$$d^2 = c^2 - c^2 \cos^2 B$$

Then, expressing the value of c^2 cosine^2 B by the formula of section 57, and observing: that c^2, being found both in the numerator and denominator, compensates itself; we have:

$$d^2 = c^2 - \left(\frac{c^2 + a^2 - b^2}{2a}\right)^2$$

And thence:

$$d^2 = \frac{4\,a^2\,c^2 - (c^2 + a^2 - b^2)^2}{4\,a^2}$$

The two terms of the numerator, being the difference of the

two squares, may be expressed by the product of the sum and difference of their roots; whence:

$$d^2 = \frac{(2\,ca + c^2 + a^2 - b^2)\,(2\,ca - c^2 - a^2 + b^2)}{4\,a^2}$$

$$= \frac{\left((c+a)^2 - b^2\right)\left(b^2 - (c \backsim a)^2\right)}{4a^2}$$

Applying the same principle again to the two factors of the numerator, we obtain:

$$d^2 = \frac{(c + a + b)\,(c + a - b)\,(c + b - a)\,(b + a - c)}{4\,a^2}$$

Then extracting the root:

$$d = \frac{1}{a}\left(\frac{(c + a + b)\,(c + a - b)\,(c + b - a)\,(b + a - c)}{4}\right)^{\frac{1}{2}}$$

This value of the perpendicular, being multiplied by half the base, $\frac{a}{2}$, gives the Surface:

$$S = \frac{da}{2}$$

$$S = \tfrac{1}{2}\left(\frac{(c + a + b)\,(c + a - b)\,(c + b - a)\,(b + a - c)}{4}\right)^{\frac{1}{2}}$$

Bringing the $\frac{1}{2}$ under the radical, by squaring it, and distributing the fourth power of 2, which is produced by this insertion, among the four factors of the numerator, the final formula becomes:

$$S = \left(\frac{a + b + c}{2} \cdot \frac{c + a - b}{2} \cdot \frac{c + b - a}{2} \cdot \frac{a + b - c}{2}\right)^{\frac{1}{2}} \qquad 4$$

Which, by the introduction of $p = \dfrac{a + b + c}{2}$, as in section 57, becomes:

5 $$S = (p\,(p - a)\,(p - b)\,(p - c))^{\frac{1}{2}}$$

§ 64. *Problem* 4. Given, two sides, and the angle opposite to one of them; to find the Surface.

Given, b, c, and C; to find S.

In figure 8, we have the perpendicular $BD' = a$ sine $C = d'$; and, according to general principles,

$$S = \frac{bd'}{2} = \frac{ba \sin C}{2}.$$

Substituting for a the value given by section 58, formula 19, we obtain:

6 $$S = \frac{b^2 \cos C \sin C}{2} \pm \frac{c \,.\, b \,.\, \sin C}{2}\left(1 - \frac{b^2 \sin^2 C}{c^2}\right)^{\frac{1}{2}}$$

This formula evidently labours under the same disadvantages for calculation as Y, No. 19; therefore the remarks made there, equally apply here; that is, determining first B by Y, No. 1, and then S by Z, No. 1, will be preferable to this formula, even with the reductions that might be made in it.

§ 65. The four preceding solutions resolve the problem for every possible case of data; they complete the full series of solutions that may be required for a triangle with simple data.

It may be conceived, that it is possible, that, instead of the quantities themselves, it may happen that the sum and difference of some of them may be given, or some other relation between the parts. But it does not belong to an elementary system of Trigonometry to enter into these details. The principles here laid down show what parts must be determinable from the data, in order that a solution may be

obtained; they also show, conversely, what data will serve to determine a triangle in complicated cases.

What remains to be said upon this subject, therefore, belongs properly to a further extension of the application of Trigonometry to the solutions of the problems that relate to triangles; and, consequently to all rectilineal figures, since Geometry teaches the means of decomposing them all into triangles.

It may, perhaps, be proper in this place to call the attention of the reader to the general character of the formulæ of series Z, analogous to what is said in section 59; namely: that they all present two lineal dimensions, with certain trigonometric functions as factors, which represent, as stated in section 11, mere numbers; that is, relations of quantities. This evidently characterizes them as giving in result a quantity of two dimensions, that is, a surface. It will easily be observed, that formula 4, having four lineal dimensions under the radical, presents also in its final result only two dimensions, by the extraction of the root.

PART III.

SPHERICAL TRIGONOMETRY.

CHAPTER I.

Introduction.—Lemmata, and Definitions of Spherical Trigonometry.

§ 66. We know from the elements of Geometry: that the circle is generated by the revolution of a finite straight line around one of its extremities; and that the sphere is generated by the revolution of the circle around one of its diameters.

Then, because all the radii of the circle are equal, all the radii of the sphere are equal also, and all the circles that pass through the centre of the sphere, may be considered as generating circles; they are also the largest circles that can be described in a sphere, for which reason, they are called *Great Circles*.

§ 67. All the other circles of the sphere, that do not pass through its centre, are *Less Circles*, and are so called; they are also called *Parallels*, because they are necessarily parallel to some one great circle of the sphere; the consideration of their properties forms no part of elementary Spherical Trigonometry. It is only great circles, then, that are objects of Spherical Trigonometry.

§ 68. *Proposition.* Three straight lines that meet in one point, and do not lie in the same plane, determine a spherical

triangle; the angle formed by any two of these lines, at the point of intersection, is the measure of the corresponding side of the spherical triangle; and the inclination of any two of the planes, that pass through each pair of the lines, is the measure of the corresponding angle of the spherical triangle; and the intersections of these planes with the surface of the sphere, describe upon it three arcs of great circles, forming a spherical triangle.

Demonstration. In figure 9, AB, AB', AB'' being three straight lines, in different planes, and A their common point of intersection; if the centre of a sphere be placed at A, the lines AB, AB', AB'', or their prolongations, will determine three points on the surface of the sphere, as C, C', C'', for we have $AC = AC' = AC''$; and the arcs CC', $C'C''$, CC'', are parts of the circumferences of the great circles passing through the planes CAC', $C'AC''$, and CAC'', respectively, and determined by the lines AC, AC'; AC', AC''; and AC, AC''; a spherical triangle, $CC'C''$, will therefore be determined upon the surface of the sphere, by the intersection of this surface with the sides of the triangular pyramid whose summit is at the centre of the sphere. The inclination of the planes that pass through each pair of these lines, just mentioned, are (Euclid, Book xi. Def. 6) measured by the angles made by the perpendiculars erected in each plane from some point in their common intersection. The arcs CC', $C'C''$, CC'', as well as their respective tangents, are perpendicular to this common intersection, each in its plane respectively; therefore the inclinations of these planes are the measure of the corresponding angles of the spherical triangle.

We therefore have, in any spherical triangle, $CC'C''$, the following results, viz:

$$\text{arc } CC' = \text{angle } BAB'$$
$$\text{arc } C'C'' = \text{angle } B'AB'$$
$$\text{arc } CC'' = \text{angle } BAB''$$

spheric angle $CC'C''$ = angle of inclination of BAB', and $B'AB''$;
" $C'CC''$ = " " $B'AB$, " BAB'';
" $CC''C$ = " " BAB', " $B'AB$.

From what has been stated, we learn:

1*st.* That all the parts of a spherical triangle, whether angles or sides, may be expressed by the trigonometric functions, that have constituted the object of the investigations of the First Part of this treatise; for they are all angles, or arcs, the measures of angles.

2*d.* That the principles of Solid Geometry, that are applicable to the sphere and triangular pyramid, are also the first principles of Spherical Trigonometry.

§ 69. Pursuing the application of Solid Geometry, we obtain a series of lemmata, to be used as fundamental principles of Spherical Trigonometry.

Lemma 1. Two great circles of a sphere cannot cut one another, except in one of their diameters; because their line of intersection must pass through the centre of the sphere, which is a point common to them all. The arcs described upon the surface of the sphere, between these intersections, will therefore be equal to two right angles, or 180°.

Lemma 2. If a line be drawn through the centre of the sphere perpendicular to any two of the lines, as AB'' and AB', this line will be a perpendicular (or *normal*) to the plane passing through these lines; and consequently to the great circle of which $C'C''$ is a portion; and all the arcs on the surface of the sphere, intercepted between this perpendicular and the great circle passing through $C'C''$, will be = 90°; or represented by a right angle at the centre of the sphere A. (Euclid, B. xi. Prop. 18.)

As this is the case on both sides of the great circle passing through $C'C''$, two points are thus determined by it on the surface of the sphere, that are called the poles, P, p, *figure* 10, of the great circle passing through $C'C''$.

We have thence:

$PC' = PC'' = PD = pC' = pC'' = pD = \llcorner R = 90°$

And the same is true of any arc whatever, drawn from P, or p, to the great circle, $DC'C''$.

Lemma 3. Every plane that passes through the line PAp, is perpendicular to this circle; and hence, every great circle of the sphere passing through the poles, Pp, is perpendicular to the circle, $DC'C''$, since these circles are intersections of the planes passing through Pp, and the surface of the sphere; and, conversely, every great circle perpendicular to the circle through $DC'C''$, passes through the poles, P, p. (Euclid, B. xi. Prop. 18.)

Lemma 4. The plane that passes through the tangents, of two great circles, that cut each other in the point P, or p, *figure* 10, is parallel to the great circle of which these points are the poles; for the diameter PAp, is perpendicular to both these planes. (Euclid, B. xi. Prop. 14.)

Lemma 5. The angles which two arcs, such as PC' and PC'', *figure* 10, make at the pole of a great circle, is equal to the arc, $C'C''$, of the great circle intercepted by these arcs; for it is the measure of the angle of inclination of their planes; and the angles at the two poles are equal; since they are measured by the same arc, the same circumstances taking place at both poles. (Euclid, B. xi. Prop. 10.)

Lemma 6. The angle subtended by the poles of two great circles is equal to the angle of inclination of the planes of the two circles of which they are the poles; for these two circles are perpendicular to their axes that pass through the poles. Or, in *figure* 11, the arc Pp is equal to the arc BC, which measures the inclination of these planes, and lies in the same plane with Pp; because AP, Ap, AB, AC, are all perpendicular to the common intersection, Dd, of the two circles, (Euclid, B. xi. Prop. 5) and PAB, and pAC, are both right angles, having the part pAB common; therefore, their complements are equal, or $Pp = BC$.

Lemma 7. If an arc of a great circle fall upon another arc of a great circle, the sum of the two angles, which it

makes with this arc, on the two sides, are equal to two right angles; for they are measured by the angles of the tangents to these arcs. And the sum of all the angles made by any number of arcs of great circles, cutting one another at the same point, is always equal to four right angles; and also, the vertical angles made by two arcs, intersecting each other, are always equal; for they are all measured by the tangents to these arcs, and these tangents lie in one plane, being all perpendicular to the radius of the sphere; and thus the Propositions, Euclid, B. i. Prop. 13 and 15, become applicable.

Lemma 8. The perpendicular arc, *Pcd*, *figure* 10, drawn through one of the angular points of the spherical triangle to the opposite side, is a part of a great circle, passing through this angular point, and the poles of the great circle, of which the opposite side is a portion; for no other plane, than one passing through its poles, can be perpendicular to a great circle.

Lemma 9. Two great circles that cut each other, enclose between them a portion of the spherical surface, that has the same ratio to the surface of the sphere, that the angle of their inclination has to four right angles; for the surface of this section is measured by (or proportional to) the angle of the inclination of the planes of these great circles; and the whole circumference is measured by four right angles; that is to say: as we have the circumference of the great circle $= 2\,r\,\pi$, the whole surface of the sphere is represented, according to these considerations, by

$$S = 2 . r^2 . \pi . 4 . \llcorner R$$

And any section of the surface, measured by the angle a, is

$$s = 2 . r^2 . \pi . a$$

whence $\quad S : s = 4 \llcorner R : a.$

§ 70. All the propositions of elementary geometry, that determine the data that are necessary to infer the equ and proportionality between the parts or surfaces of

triangles, are true of spherical triangles; except, that the exterior angle of a spherical triangle is not equal to the sum of the two interior and opposite angles, because the three angles of the spherical triangle do not lie in the same plane; for this reason, the knowledge of two angles of a spherical triangle does not imply the determination of the magnitude of the third, as in plane trigonometry; while, on the other hand, if any three parts of a spherical triangle be given, the remaining three may be determined, even although no one of the parts given be a side. This last follows from the circumstance, that all the quantities concerned in a spherical triangle are of the same nature. All these are evident consequences of the preceding sections, which show that the spheric triangle is the intersection of the triangular pyramid with the surface of the sphere; all that would apply to the plane triangular base of the pyramid must therefore also be true for the spheric base, with the exception that has been stated, which is evidently a consequence of what has been already said.

§ 71. *Theorem.* The sum of all the sides of a spherical triangle is always less than four right angles; and any one side is always less than the sum of the other two sides.

Demonstration. The three planes that intersect each other in the lines AC, AC', AC'', *figure* 9, form at the point A, or the centre of the sphere, a solid angle; the sum of all the plane angles forming a solid angle around a point, is always less than four right angles. (Euclid, B. xi. Prop. 21.) As the three sides of the triangle formed upon the surface of the sphere, by the intersection of these planes, are the measures of the angles at the centre, (section 68,) their sum is also less than four right angles. For the same reason it follows (from Euclid, B. xi. Prop. 20) that the sum of any two of the sides is greater than the third side.

§ 72. *Theorem.* The sum of the three angles of a spherical triangle is always less than six, and more than four right angles.

Demonstration. The sum of the angles forming the three

solid angles, at the three angular points of the spherical triangle enclosed by the planes passing through its sides and the plane through the tangents at the sphere, (or the spherical surface, as these angles are the same,) is less than three times four right angles, (or twelve right angles,) for each of them is less than four right angles. (Euc. B. xi. Pr. 21.) But the angles formed by the tangents and the intersections of the planes, are right angles, and their sum is three times two, or six, right angles. The sum of the remaining three angles, formed by the tangents, which are the same with the spherical angles, is therefore less than six right angles.

Or, more briefly and algebraically :

Calling S = sum of the angles forming the 3 solid angles;
" S' = sum of the angles of the triangle, or of the tangents; we have :

$$S = S' + 6 \llcorner R$$

$$S \angle 12 . \llcorner R$$

therefore $$S' + 6 \llcorner R \angle 12 \llcorner R$$

Subtracting $6 \llcorner R$ from both sides :

$$S' \angle 6 \llcorner R$$

Q : E : D

If the three angles were equal together to two right angles, the three sides would be in the same plane; the lines of intersection of the planes perpendicular to the tangents would be parallel to each other, and would no longer meet in the centre of the sphere, which is contrary to the supposition; therefore, the sum of the spherical angles must be *more* than four right angles.

§ 73. *Theorem.* If, from the three angular points of a triangle, arcs be drawn on the surface of the sphere, whose distances from the angular points are each = 90°, the intersections of these arcs will form a new triangle, that is called supplementary; that is to say: the sides of this new triangle

will be the supplements of those angles of the original triangle, that are opposite to them; and the angles of the new triangle will be the supplements of the sides of the original triangle, which are opposite to them. Moreover, the angular points of the original triangle will be the poles of the sides of the new triangle; and the angular points of this, will be the poles of the sides of the original triangle. They, therefore, are also called polar triangles, in respect to each other.

Construction. In *figure* 12, let *ABC* be the spherical triangle from whose angular points the arcs *DE, EF, FD*, are drawn, at the distance of 90°; the points *A, B, C*, will be respectively, the poles of the arcs that cut each other in the points *D, E, F*; and form the supplementary triangle *DEF*.

Produce the sides of the original triangle, *ABC*, until they intersect the arcs *DE, FE, FD*, in the points *G, H, L, M, J, K*.

Demonstration. Because the point *E* is 90° distant from each of the two points *A*, and *B*, this point *E*, is the pole of the circle *LABG*, that passes through the points *A* and *B*; for the same reason, the point *F* is the pole of the arc *KBCM*; and the point *D*, the pole of the arc *JACH*; whence we have:

$$DH + EG = DE + GH = 2 \llcorner R = 180°$$

But, because *GH* is an arc of a circle distant 90° from the point *A*, it is the measure of the spherical angle at *A*, formed by the two arcs *AG, AH*; and *DE* is the side of the supplementary triangle that is opposite to the angle *BAC*; for which reasons we have:

$$DE = 180° - GH = 180° - A.$$

In like manner we obtain:

$$EF = 180° - LM = 180° - B.$$

and $$DF = 180° - JK = 180° - C.$$

Moreover, since the arc $EL = EG = 90°$; and *E* is the

pole of the arc $GBAL$, or AB, produced until it intersects the other arcs EF, and ED; the arc GL is the measure of the angle at E, of the supplementary triangle; whence we have:

$$LB + AG = LG + AB = 2 \llcorner R = 180^\circ.$$
$$\text{therefore} \qquad LG = 180^\circ - AB.$$

AB, then, is the supplement of LG; or of the angle E, which it measures, or is equal to. For the same reason, we have for all the angles of the supplementary triangle, expressed by the sides of the original triangle:

$$LG = E = 180^\circ - AB.$$
$$HG = D = 180^\circ - AC.$$
$$KM = F = 180^\circ - BC.$$

Therefore, generally, the sides and angles of the supplementary triangle are the supplements of the angles and sides of the original triangle.

§ 74. *Theorem.* The surface of a spherical triangle is proportional to the spherical excess of its three angles above two right angles.

Construction. Let abc, *figure* 13, be a spherical triangle. Produce ac, until the entire circumference of the great circle $aced$, be completed. Produce also ab, and bc, until they cut this great circle, in the points d, and e, and also to their common intersection, in f, on the opposite side of the great circle $cade$, which point f, will be the opposite pole to the point b. (*Lemma* 1.)

Demonstration. By construction, $af = bc$, as they are each of them a supplement of the arc ab; for

$$abe = 2 \llcorner R = baf;$$

for the same reason, the arc $cf = bd$; and the angles at f, and b, are equal; for they are at the opposite poles, and between the same arcs; wherefore the triangles bde, and fac, are equal.

The hemisphere whose base is the great circle *cade*, is, *by lemma* 9, equal or proportional to (calling H = hemisphere)

$$H = 2 . r^2 . \pi \, 2 \llcorner R$$

The section of the sphere between the two semicircles *baf*, *bcf*, is equal, or proportional to:

$$abc + acf = abc + deb = 2 \, r^2 \, \pi \, b$$

because the angle b is the measure of the inclination of the planes of these two circles.

For the same reason we have:

The spheric section; $dbcad = abc + abd = 2 \, \pi \, r^2 . c$

" " $ebace = abc + cbe = 2 \, \pi \, r^2 . a$

Subtracting from the sum of all these, twice the value of the triangle *abc*, which is contained in each of them, and of course three times in their sum, the whole hemisphere is represented thus:

$$H = daced = 2 . r^2 \pi \llcorner R$$
$$= 2 . r^2 \, \pi \, b + 2 . r^2 \, \pi \, c + 2 \, r^2 \, \pi \, a - 2 \, abc$$

whence

$$2 . r^2 \, \pi \, 2 . \llcorner R = 2 . r^2 \, \pi \, (b + c + d) - 2 . abc$$

or

$$r^2 \, \pi . 2 \llcorner R = r^2 \, \pi \, (b + c + a) - abc$$

And transposing:

$$abc = \pi \, r^2 \, (a + b + c) - 2 \, \pi \, r^2 \llcorner R$$
$$= \pi \, r^2 \, (a + b + c - 2 \llcorner R)$$

The surface of the spherical triangle is equal to the product of the square of the radius of the sphere into the excess of the three spherical angles above two right angles; and as we always assume the radius = 1, (until applied to a determinate sphere,) we have:

$$abc = a + b + c - 2 \llcorner R$$

CHAPTER II.

Investigation of the Fundamental Formulæ of Spherical Trigonometry; and Solutions of Right Angled Spherical Triangles.

§ 75. *General Problem.* To determine the relations between the sides and angles of a right angled spherical triangle.

Let *DA*, *DB*, *DC*, *figure* 14, be three lines, that are not in the same plane, and which, by their intersection at the point *D*; determine the spherical triangle *ABC*; and let the plane *BDA*, be perpendicular to the plane *ADC*; these, by the preceding chapter, are the elements of a spherical triangle, *ABC*, right angled at A.

Construction. In the line *DB*, take any point, *E*, and from it draw *EG*, perpendicular to *DC*; and from *G*, where *EG* intersects the *DC*, draw, in the plane *ADC*, *GF*, perpendicular to *DC*; join *EF*; the angle *EGF* will be the angle of inclination of the planes *BDC*, and *ADC*, and therefore equal to the spherical angle *BCA*.

Solution. By construction, *CD* is perpendicular to the plane passing through *EGF*; for a like reason, the plane of *DGF*, that passes through this perpendicular, is perpendicular to the plane *EGF*; and by supposition, the plane *BDA*, is perpendicular to the plane *CDA*; therefore the two planes *BDA*, and *EGF*, are perpendicular upon *CDA*, and their common intersection, *EF*, is also perpendicular to *CDA*; and the angles *EFD*, and *EFG*, are right angles; and the four triangles, *DGE*, *DGF*, *DFE*, and *GFE*, are all right angled, at the points *G*, and *F*. (Euc. B. xi. Prop. 4, 18, 19.)

Each pair of these triangles has one side common, and one angle (a right angle) equal in each; wherefore, if two sides, in any one of the triangles, be given, two sides in an other, may

be determined; and also: the determination of two of the triangles will afford the means of determining a third.

Determining, then, upon the principles of section 7, the various relations of the parts of these triangles; keeping it constantly in mind: that the *ratios* of the sides of a right angled triangle, and not their absolute values, determine the trigonometric functions; if we adopt, for the sake of brevity, the following notation:

$$EF = c'; \quad EG = a'; \quad GF = b';$$
$$DE = g; \quad DF = e; \quad DG = f;$$

And for the arcs; $BC = a$; $AB = c$; $AC = b$;

we have as follows:

1. Determining the triangle *EGF*, by means of the triangles *DEF*, and *DEG*:

$$\frac{EF}{ED} = \frac{c'}{g} = \sin BDA = \sin BA = \sin c$$

and
$$\frac{EG}{ED} = \frac{a'}{g} = \sin BDC = \sin BC = \sin a$$

And by division, (using only the denominations adopted above:)

$$\frac{c'}{g} : \frac{a'}{g} = \frac{c'}{a'} = \frac{\sin c}{\sin a}$$

In the triangle *EFG* we have also (according to section 68:)

$$\frac{EF}{EG} = \frac{c'}{a'} = \sin EGF = \sin C$$

From these two results we obtain the following equation:

a $$\frac{c'}{a'} = \frac{\sin c}{\sin a} = \sin C$$

1 or $$\sin c = \sin C \sin a$$

2. **Determining *DGF*, by means of *DEF*, and *DEG*, we have:**

$$\frac{DF}{DE} = \frac{e}{g} = \cos BDA = \cos BA = \cos c$$

and $$\frac{DG}{DE} = \frac{f}{g} = \cos BDC = \cos BC = \cos a$$

Dividing the second by the first:

$$\frac{f}{g} : \frac{e}{g} = \frac{f}{e} = \frac{\cos a}{\cos c}$$

And by the triangle *DGF*:

$$\frac{DG}{DF} = \frac{f}{e} = \cos ADC = \cos AC = \cos b$$

And by equality from these two results:

$$\frac{f}{e} = \frac{\cos a}{\cos c} = \cos b$$

or $$\cos a = \cos b \cos c \qquad 2$$

3. **Determining *DGF*, by *EGF*, and *EGD*, we obtain:**

$$\frac{EG}{DG} = \frac{a'}{f} = \tan BDC = \tan BC = \tan a$$

and $$\frac{FG}{DG} = \frac{b'}{f} = \tan ADC = \tan AC = \tan b$$

And by division of the second by the first:

$$\frac{b'}{f} : \frac{a'}{f} = \frac{b'}{a'} = \frac{\tan b}{\tan a}$$

By the triangle *DGF*:

$$\frac{b'}{a'} = \cos EGF = \cos C$$

Hence by equality:

$$\frac{b'}{a'} = \frac{\tan b}{\tan a} = \cos C$$

3 or $\tan b = \tan a \cos C$

4. Determining EGF, by DGF, and DEF:

$$\frac{EF}{DF} = \frac{c'}{e} = \tan BDA = \tan BA = \tan c$$

and $$\frac{GF}{DF} = \frac{b'}{e} = \sin ADC = \sin AC = \sin b$$

And by division:

$$\frac{c'}{e} : \frac{b'}{e} = \frac{c'}{b'} = \frac{\tan c}{\sin b}$$

By the triangle EGF:

$$\frac{c'}{b'} = \tan EGF = \tan C$$

Whence by equality:

$$\frac{c'}{b'} = \frac{\tan c}{\sin b} = \tan C$$

4 or $\tan c = \tan C \sin b$

§ 76. If we substitute for the angle C, in such of the preceding formulæ as contain it, the angle B, and for the sides of the triangle, those which have the same position in relation to the angle B, as the sides named in the preceding formulæ, have in relation to the angle C; the formulæ No. 1, 3, and 4, will give also the following:

From No. 1, is obtained; $\sin a \sin B = \sin b$ 5

3, $\cos B \tan a = \tan c$ 6

4, $\sin c \tan B = \tan b$ 7

The formula No. 2 cannot be changed, because it always expresses the relation of equality that exists between the cosine of the hypothenuse, and the product of the cosines of the sides that contain the right angle.

The comparison of the formulæ already obtained in this chapter, gives, by means of the equalities that are found in them, the following formulæ, which complete all the cases of right angled spherical trigonometry.

Comparing No. 4 with No. 6:

$$\tan c = \sin b \tan C = \cos B \tan a$$

Substituting from No. 5: $\sin b = \sin a \sin B$

$$\sin a \sin B \tan C = \cos B \tan a$$

Dividing by: $\sin B \tan a$:

$$\cos a \tan C = \cot B$$ 8

By substituting, in the same equation, from No. 3, $\tan a = \dfrac{\tan b}{\cos C}$, the equation becomes:

$$\sin b \tan C = \frac{\cos B \tan b}{\cos C}$$

Multiplying by $\cos C \cot b$:

$$\cos b \sin C = \cos B.$$ 9

§ 77 The nine formulæ of the two preceding sections may be reduced to six, by remarking: that No. 5, 6, and 7, are mere repetitions of No. 1, 3, and 4; for they differ only in having relation to the other oblique angle of the triangle. They give all the cases of right angled spherical trigonometry. It may be of use to repeat these principal formulæ, and to

transform them into other similar ones, applicable to a triangle that has one side $= 90°$, by means of the supplementary triangle, whose properties have been explained in section 73.
b We have then:

1 $\sin c = \sin a \sin C$

2 $\cos a = \cos b \cos c$

3 $\tan b = \tan a \cos C$

4 $\tan c = \sin b \tan C$

5 $\cot B = \cos a \tan C$

6 $\cos B = \cos b \sin C$

Transforming these equations to a triangle, one of whose sides is $= 90°$, by means of the supplementary triangle, and using the same characteristic characters, merely accentuated, we have:

7 $\sin C' = \sin A' \sin c'$

8 $\cos A' = \cos B' \cos C'$

9 $\tan B' = \tan A' \cos c'$

10 $\tan C' = \sin B' \tan c'$

11 $\cot b' = \cos A' \tan c'$

12 $\cos b' = \cos B' \sin c'$

The signs of the trigonometric functions have no influence in these mutations.

CHAPTER III.

Investigation of the General Formulæ of Oblique Angled Spherical Trigonometry.

§ 78. *General Problem.* To find the relations between the parts of an oblique angled spherical triangle.

Construction. Let *ABC, figures* 15 and 16, be a spherical triangle, that has either three acute angles, or the angle at *A*, obtuse.

From *C*, draw the arc *CD*, perpendicular to *BA*; it will in the first case, be opposite to the two interior angles *A*, and *B*; and in the second, to the interior angle at *B*, and the exterior at *A*, or to the supplement of the interior angle *A*; the functions of which exterior angle are the same with those of the interior angle *A*, of the triangle.

Solution. The two right angled triangles, *CBD*, and *CAD*, formed by the perpendicular *CD*, have this perpendicular common to both. If, then, we express any one of the functions of this side by the functions of the other parts of each of the triangles, we shall obtain equations between the functions of these parts, which may be transformed into proportions, and which will furnish all the solutions of the oblique angled spherical triangle *ABC*, by means of its parts.

Performing this process for the sine, the cosine, and the tangent of *CD*, we obtain the following table; which follows as the direct consequence of the formulæ b, No. 1 to 6, of section 77. We have therefore :

	By the triangle *BCD* :	By the triangle *ACD* :	c
$\sin CD$	$= \sin a \,.\, \sin B$	$= \sin b \,.\, \sin A$	1
	$= \tan BD : \tan BCD$	$= \tan AD : \tan ACD$	2
$\cos CD$	$= \cos a : \cos BD$	$= \cos b : \cos AD$	
	$= \cos B : \sin BCD$	$= \cos A : \sin ACD$	
$\tan CD$	$= \tan a \,.\, \cos BCD$	$= \tan$ [illegible] ACD	
	$= \sin BD \,.\, \tan B$	$=$ [illegible]	

As the first formula contains no other terms than the parts of an oblique angled triangle, it is evident that its result is general; for an analogous result would be obtained, if the perpendicular were let fall from one of the other angles of the triangle, upon its opposite side; we therefore have likewise:

7 $$\sin c \sin B = \sin C \sin b$$

8 $$\sin C \sin a = \sin A \sin c$$

These formulæ, (1, 7, and 8,) transformed into proportions, give:

9 $$\sin a : \sin b = \sin A : \sin B$$

10 $$\sin c : \sin b = \sin C : \sin B$$

11 $$\sin a : \sin c = \sin A : \sin C$$

By which it appears that we have, in Spherical Trigonometry, a general principle analogous to that found in Plane Trigonometry; viz: that *the sines of the sides are proportional to the sines of the opposite angles.*

The formulæ 5 and 6, transformed into proportions, give the following results, viz:

12. From No. 5; $\tan a : \tan b = \cos ACD : \cos BCD$

13 " 6; $\tan A : \tan B = \sin BD : \sin AD$

Corollary. The sines of the perpendiculars let fall from the different angular points upon the opposite sides in a spherical triangle, are to one another inversely as the sines of those sides.

Demonstration. We had by c, No. 1, (*figure* 17:)

$$\sin CD = \sin a \sin B$$

Drawing from the point A, an arc perpendicular to BC, we have from the same principles:

$$\sin AD' = \sin c \sin B$$

therefore

14 $$\sin CD : \sin AD' = \sin a : \sin c$$

§ 79. Considering the formulæ c, No. 2, 3, 4, 9, 12, and 13, as proportions; and combining them by addition and subtraction, as allowed by the principles of proportion; we deduce a series of formulæ, useful in the transformations of trigonometric formulæ, to adapt them for calculation. They also form, in conjunction with the original proportions from which they are deduced, a series of solutions of a spherical triangle, by means of its parts, formed by a perpendicular drawn from one of its angles on the opposite side. When the angle is obtuse, it is known by the algebraic sign of the trigonometric functions.

By this operation we obtain the following results in succession, in which we adopt for the statement of proportions, the mode of writing them as equations of fractions; (as more convenient;) and, to render the process more easy, we assume the following notations, in addition to those already mentioned; viz:

$$AD = c_{,} \; ; \qquad BD = c_{,,} \; ;$$
$$ACD = C_{,} \; ; \qquad BCD = C_{,,}$$

whence

$$AD + BD = c_{,} + c_{,,} = c$$
$$ACD + BCD = C_{,} + C_{,,} = C$$

We have by c, No. 9:

$$\frac{\sin a}{\sin b} = \frac{\sin A}{\sin B}$$

Adding and subtracting this proportion by the well known method, we have:

$$\frac{\sin a + \sin b}{\sin a \backsim \sin b} = \frac{\sin A + \sin B}{\sin A \backsim \sin B}$$

And substituting from series M, No. 7 and 10:

$$\frac{\sin \frac{1}{2}(a+b)\cos \frac{1}{2}(a \backsim b)}{\cos \frac{1}{2}(a+b)\sin \frac{1}{2}(a \backsim b)} = \frac{\sin \frac{1}{2}(A+B)\cos \frac{1}{2}(A \backsim B)}{\cos \frac{1}{2}(A+B)\sin \frac{1}{2}(A \backsim B)}$$

d or

1
$$\frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}(a \backsim b)} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A \backsim B)}$$

From c, No. 2, is obtained by the same means:

$$\frac{\tan c_{,,} + \tan c_{,}}{\tan c_{,,} \backsim \tan c_{,}} = \frac{\tan C_{,,} + \tan C_{,}}{\tan C_{,,} \backsim \tan C_{,}}$$

Substituting from G, No. 1:

$$\frac{\sin (c_{,,} + c_{,})}{\sin (c_{,,} \backsim c_{,})} = \frac{\sin (C_{,,} + C_{,})}{\sin (C_{,,} \backsim C_{,})}$$

or

2
$$\frac{\sin c}{\sin (c_{,,} \backsim c_{,})} = \frac{\sin C}{\sin (C_{,,} \backsim C_{,})}$$

From c, No. 3, is obtained by this method:

$$\frac{\cos a + \cos b}{\cos a \backsim \cos b} = \frac{\cos c_{,,} + \cos c_{,}}{\cos c_{,,} \backsim \cos c_{,,}}$$

Substituting from series M, No. 8 and 11:

$$\frac{\cos \frac{1}{2}(a+b) \cos \frac{1}{2}(a \backsim b)}{\sin \frac{1}{2}(a+b) \sin \frac{1}{2}(a \backsim b)} = \frac{\cos \frac{1}{2} c \cos \frac{1}{2}(c_{,,} \backsim c_{,})}{\sin \frac{1}{2} c \sin \frac{1}{2}(c_{,,} \backsim c_{,})}$$

or

3
$$\frac{\cot \frac{1}{2}(a+b)}{\tan \frac{1}{2}(a \backsim b)} = \frac{\cot \frac{1}{2} c}{\tan \frac{1}{2}(c_{,,} \backsim c_{,})}$$

From c, No. 4, is obtained by this method:

$$\frac{\cos B + \cos A}{\cos B \backsim \cos A} = \frac{\sin C_{,,} + \sin C_{,}}{\sin C_{,,} \backsim \sin C_{,}}$$

And substituting from M, No. 7, 8, 10, and 11:

$$\frac{\cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A \backsim B)}{\sin \frac{1}{2}(A+B) \sin \frac{1}{2}(A \backsim B)} = \frac{\sin \frac{1}{2} C \cos \frac{1}{2}(C_{,,} \backsim C_{,})}{\cos \frac{1}{2} C \sin \frac{1}{2}(C_{,,} \backsim C_{,})}$$

or

$$\frac{\cot \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A \backsim B)} = \frac{\tan \frac{1}{2} C}{\tan \frac{1}{2}(C_{,,} \backsim C_{,})} \qquad 4$$

From c, No. 12, is obtained:

$$\frac{\tan a + \tan b}{\tan a \backsim \tan b} = \frac{\cos C_{,} + \cos C_{,,}}{\cos C_{,} \backsim \cos C_{,,}}$$

Substituting from G, No. 1, and M, No. 8 and 11:

$$\frac{\sin (a+b)}{\sin (a \backsim b)} = \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2}(C_{,,} \backsim C_{,})} \qquad 5$$

From c, No. 13, is obtained in a similar manner:

$$\frac{\sin c_{,,} + \sin c_{,}}{\sin c_{,,} \backsim \sin c_{,}} = \frac{\tan A + \tan B}{\tan A \backsim \tan B}$$

Substituting from M, No. 7 and 10, and G, No. 1:

$$\frac{\sin \frac{1}{2} c \cos \frac{1}{2}(c_{,,} \backsim c_{,})}{\cos \frac{1}{2} c \sin \frac{1}{2}(c_{,} \backsim c_{,,})} = \frac{\sin (A+B)}{\sin (A \backsim B)}$$

or

$$* \quad \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2}(c_{,,} \backsim c_{,})} = \frac{\sin (A+B)}{\sin (A \backsim B)} \qquad 6$$

§ 80. The section 78 has already given one of the general principles for the solution of Spheric Trigonometry, applicable to the cases, where two sides and one angle, or two angles and one side are given; so that one of the given angles and sides are opposite to each other, and the part sought opposite to the remaining part given.

To obtain other such formulæ, in which the three parts given are two sides and the included angle, and the part

* In these formulæ and their combinations lie all those called Napier's analogies, which it is not here necessary to treat in full.

sought one of the other angles; or when two angles and the included side are given, and one of the other sides sought; we have not here given, as in Plane Trigonometry, the sum of the remaining angles. A formula that is applicable to this case, and to those derived from it, may be obtained in the following manner:

We have from a, No. 3, by transforming the denominations into those here employed:

$$\tan c_{\prime\prime} = \tan a \cos B$$

By F, No. 1:

$$\sin c_{\prime} = \sin (c \backsim c_{\prime\prime}) = \sin c \cos c_{\prime\prime} \backsim \cos c \sin c_{\prime\prime}$$

Whence

$$\frac{\sin c_{\prime}}{\sin c_{\prime\prime}} = \frac{\sin c}{\tan c_{\prime\prime}} \backsim \cos c$$

By c, No. 13:

$$\frac{\sin c_{\prime}}{\sin c_{\prime\prime}} = \frac{\tan B}{\tan A}$$

Therefore, by equality:

$$\frac{\tan B}{\tan A} = \frac{\sin c}{\tan c_{\prime\prime}} \backsim \cos c$$

Substituting for tangent $c_{\prime\prime}$ the first of the above formulæ:

$$\frac{\tan B}{\tan A} = \frac{\sin c}{\tan a \cos B} \backsim \cos c$$

Reducing to a common denominator gives, after compensating this common denominator on both sides:

$$\tan a \sin B = \sin c \tan A \backsim \cos c \tan a \cos B \tan A$$

e Multiplying by: cotangent a cotangent A:

1 $$\cot A \sin B = \sin c \cot a \backsim \cos c \cos B$$

As this formula contains the functions of two sides and two angles, of which one is opposite and an other included, alternately; if any three of these parts be given, the fourth may be found; it is, therefore, general in all cases, as has been shown to be true of formula c, No. 1, for those cases where the sides and angles are mutually opposite.

§ 81. In order to complete all the possible *modes* of combining the six parts of a spherical triangle, there only remains to be investigated, an analytic expression that shall contain three parts of the same kind, and one of a different kind; that is, three sides and one angle, or three angles and one side. A formula of this kind may be obtained by means of a process similar to that of section 80, with merely an appropriate variation in the parts substituted.

By a No. 3, we have:

$$\tan c_{\prime\prime} = \tan b \cos A$$

By F, No. 2:

$$\cos c_{\prime} = \cos (c \backsim c_{\prime\prime}) = \cos c \cos c_{\prime\prime} + \sin c \sin c_{\prime\prime}$$

Whence

$$\frac{\cos c_{\prime}}{\cos c_{\prime\prime}} = \cos c + \sin c \tan c_{\prime\prime}$$

By c, No. 3, we have also, (substituting the notation here used:)

$$\frac{\cos c_{\prime}}{\cos c_{\prime\prime}} = \frac{\cos a}{\cos b}$$

Thence by equality:

$$\frac{\cos a}{\cos b} = \cos c + \sin c \tan c_{\prime\prime}$$

And substituting for tangent $c_{\prime\prime}$, the first formula above:

$$\frac{\cos a}{\cos b} = \cos c + \sin c \tan b \cos A$$

Q

Or finally, (multiplying by cosine b:)

2 $$\cos a = \cos c \cos b + \sin c \sin b \cos A$$

§ 82. The general formulæ c, No. 1, and e, No. 1, are of an identical nature; the sides and angles appearing in them in an even number, or symmetric. It remains for us to show, that the formula e, No. 2, just preceding, has the same property of generality, as relates to its application to sides or angles, when proper attention is paid to the consequence of the change in the assumption; this may be most easily shown by a reference to the supplementary triangle.

Transforming, therefore, e, No. 2, in conformity with the principles of the supplementary triangle, we shall have all the sides changed into angles, and the angle will become a side; the formula will thus be adapted to three angles and one side; but it must be observed: that the cosines of the supplementary angle are negative; and that the algebraic signs of the different terms undergo the change consequent to this change of sign. If the transformation be performed with regard to this circumstance, and the single term on the left hand rendered positive, as is always usual for the part sought in an equation; we shall obtain the result:

3 $$\cos A = \sin C \sin B \cos a - \cos C \cos B$$

§ 83. As the three general formulæ, series c, No. 1, and series e, No. 1 and 2, include the analytical expressions of all the cases of oblique angled spherical trigonometry; it will be proper, on account of the frequent use we shall make of them, in deducing from them others, better adapted for calculation in each individual case, to collect them here, so as to be seen at a single glance. It is evident from the above: that, taking them in the full extent of their import, they contain, in the shape of no more than three formulæ, all the solutions of oblique angled spherical trigonometry; that they, therefore, are the fundamental elements of all future investigations of the individual cases.

These three formulæ are as follows:

$$\sin a \sin B = \sin b \sin A \qquad 1$$

$$\cot A \sin B = \sin c \cot a \backsim \cos c \cos B \qquad 2$$

$$\cos a = \sin c \sin b \cos A + \cos c \cos b \qquad 3$$

§ 84. We shall not, therefore, here make any mutations of these formulæ, to adapt them to individual cases, which are, of course, contained in them analytically; but keep them in the full generality of their value and import, all their mutations according to the data of a given case being of course supposed; and proceed in the next chapter to give, for each particular case, a variety of formulæ, adapted to logarithmic calculation, presenting a proper choice, according to the nature of the data of any individual calculation.

CHAPTER IV.

Deduction of Formulæ adapted to the Logarithmic Calculation of all the cases of Oblique Angled Spherical Trigonometry.

§ 85. This part will here be treated of by means of a complete series of Problems, for each different supposition of parts given and parts sought; applying the appropriate reductions, and using, in case of need, the auxiliary angles; with the modifications of the formulæ whence they are derived; and having reference, for convenience of notation, to a spherical triangle *ABC*, *figure* 15 *or* 16, whose sides *a, b, c,* are respectively opposite to the angles *A, B, C.*

§ 86. *Problem* 1. Given, two sides and the angle opposite to one of them, to find the angle opposite to the other side.

b, c, and the angle *B*, being given, to find the angle *C.*

By c, No, 7, we have given:

$$\sin B \sin c = \sin C \sin b$$

Therefore

g $$\sin C = \frac{\sin B \sin c}{\sin b}$$

This formula is adapted to logarithmic calculation, without change, as is evident from inspection.

§ 87. *Problem* 2. Given, two angles and a side opposite to one of them, to find the side opposite to the other.

c, B, and *C,* being given, to find *b.*

By the same formula we have:

$$\sin B . \sin c = \sin C \sin b$$

Therefore

h $$\sin b = \frac{\sin B \sin c}{\sin C}$$

§ 88. *Problem* 3. Given, the three sides, to find one of the angles.

a, b, c, being given, to find *A.*

By f, No. 3, we have:

$$\cos a = \sin b \sin c \cos A + \cos b \cos c$$

By transposition and division:

$$\cos A = \frac{\cos a - \cos c \cos b}{\sin b \sin c}$$

1*st Transformation.* To transform this formula for the purpose of adapting it to logarithmic calculation; we have by Q, No. 3: $\cos A = 1 - 2 \sin^2 \frac{1}{2} A$

By which:

$$1 - 2 \sin^2 \tfrac{1}{2} A = \frac{\cos a - \cos c \cos b}{\sin c \sin b}$$

Transposing and reducing to a common denominator, with the consequent change of signs:

$$2 \sin^2 \tfrac{1}{2} A = \frac{\sin c \sin b + \cos c \cos b - \cos a}{\sin c \sin b}$$

And by the two first terms of the numerator:

$$2 \sin^2 \tfrac{1}{2} A = \frac{\cos (c \backsim b) - \cos a}{\sin c \sin b}$$

Substituting from M, No. 11, and dividing both sides of the equation by 2:

$$\sin^2 \tfrac{1}{2} A = \frac{\sin \tfrac{1}{2} \left(a + (c \backsim b)\right) \sin \tfrac{1}{2} \left(a - (c \backsim b)\right)}{\sin c \sin b}$$

Calling $p = \dfrac{a + b + c}{2}$, as in section 57:

$$\sin^2 \tfrac{1}{2} A = \frac{\sin (p - c) \sin (p - b)}{\sin c \sin b}$$

Whence:

$$\sin \tfrac{1}{2} A = \left(\frac{\sin (p - c) \sin (p - b)}{\sin c \sin b}\right)^{\frac{1}{2}} \qquad 1$$

2d Transformation. We have also by Q, No 4:

$$\cos A = 2 \cos^2 \tfrac{1}{2} A - 1$$

Substituting this, transposing the 1, and reducing to a common denominator, we have by the same process as above:

$$2 \cos^2 \tfrac{1}{2} A = \frac{\cos a - (\cos c \cos b - \sin b \sin c)}{\sin b \sin c}$$

$$= \frac{\cos a - \cos (b + c)}{\sin b \sin c}$$

$$2 \cos^2 \tfrac{1}{2} A = \frac{2 \sin \frac{1}{2} (a + b + c) \sin \frac{1}{2} (b + c - a)}{\sin b \sin c}$$

2 $$\cos \tfrac{1}{2} A = \left(\frac{\sin p \sin (p - a)}{\sin b \sin c} \right)^{\frac{1}{2}}$$

These formulæ are affected in the same way as the corresponding formulæ of Plane Trigonometry, Y, No. 14 and 15.

3d *Transformation.* We also derive from them, in the way that was there described, the following formula, which is in all cases the most exact in its results.

3 $$\tan \tfrac{1}{2} A = \frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A} = \left(\frac{\sin (p - c) \sin (p - b)}{\sin p \sin (p - a)} \right)^{\frac{1}{2}}$$

§ 89. *Problem* 4. Given, the three angles, to find one of the sides.

A, B, C, being given, to find a.

By formula e, No. 3:

$$\cos a = \frac{\cos B \cos C + \cos A}{\sin B \sin C}$$

1st *Transformation.* By Q. No. 3: $\cos a = 1 - 2 \sin^2 \frac{1}{2} a$ Substituting this value, transposing, and reducing to a common denominator:

$$2 \sin^2 \tfrac{1}{2} a = \frac{\sin B \sin C - \cos B \cos C - \cos A}{\sin B \sin C}$$

$$= \frac{- \cos (B + C) - \cos A}{\sin B \sin C}$$

By a substitution analogous to M, No. 8, and dividing both sides by 2:

$$\sin^2 \tfrac{1}{2} a = - \frac{\cos \frac{1}{2} (B + C + A) \cos \frac{1}{2} (B + C - A)}{\sin B \sin C}$$

Assuming $P = \dfrac{A + B + C}{2}$, and extracting the root:

$$\sin \tfrac{1}{2} A = \left(-\frac{\cos P \cos (P - A)}{\sin B \sin C}\right)^{\frac{1}{2}}$$

We get rid of the sign —, from the consideration: that P is necessarily an obtuse angle, since the sum of the three angles of a spherical triangle is always greater than two right angles, (by section 72;) hence its half is greater than one right angle, the cosine of which being negative, renders the result positive; we have, therefore, finally:

$$\sin \tfrac{1}{2} a = \left(\frac{-\cos P \cos (P - A)}{\sin B \sin C}\right)^{\frac{1}{2}} \qquad 1$$

2d Transformation. By substituting in the above formula $\cos a = 2 \cos^2 \frac{1}{2} a - 1$, we have:

$$2 \cos^2 \tfrac{1}{2} a = \frac{\sin B \sin C + \cos B \cos C + \cos A}{\sin B \sin C}$$

$$= \frac{\cos C + \cos (B \backsim C)}{\sin B \sin C}$$

By a substitution analogous to M, No. 8, it becomes:

$$2 \cos^2 \tfrac{1}{2} a = \frac{2 \cos \frac{1}{2} \big(A + (B \backsim C)\big) \cos \frac{1}{2} \big(A \backsim (B \backsim C)\big)}{\sin B \sin C}$$

$$\cos \tfrac{1}{2} a = \left(\frac{\cos \frac{1}{2} (A + B - C) \cos \frac{1}{2} (A + C - B)}{\sin B \sin C}\right)^{\frac{1}{2}}$$

$$\cos \tfrac{1}{2} a = \left(\frac{\cos (P - C) \cos (P - B)}{\sin B \sin C}\right)^{\frac{1}{2}} \qquad 2$$

3d Transformation. We have also in this case as in the preceding, from the division of the two formulæ:

3 $$\tan \tfrac{1}{2} a = \left(\frac{\cos P \cos (P - A)}{\cos (P - C) \cos (P - B)}\right)^{\frac{1}{2}}$$

§ 90. *Problem* 5. Given, two sides and the included angle, to find the third side.

b, *c*, and *A*, being given, to find *a*.

We have found by section 88, problem 3:

$$2 \sin^2 \tfrac{1}{2} A = \frac{\cos (c \backsim b) - \cos a}{\sin b \sin c}$$

Or

$$2 \sin b \sin c \sin^2 \tfrac{1}{2} A = \cos (c \backsim b) - \cos a$$

1st Transformation. Substituting for the two cosines on the right, their values from the analogy of Q, No. 3, we have:

$$2 \sin b \sin c \sin^2 \tfrac{1}{2} A = 2 \sin^2 \tfrac{1}{2} a - 2 \sin^2 \tfrac{1}{2} (c \backsim b)$$

Whence

$$\sin^2 \tfrac{1}{2} a = \sin^2 \tfrac{1}{2} (c \backsim b) + \sin b \sin c \sin^2 \tfrac{1}{2} A$$

Making the first term on the right a common factor for both terms, and extracting the square root:

1 $$\sin \tfrac{1}{2} a = \sin \tfrac{1}{2} (c \backsim b) \left(1 + \frac{\sin b \sin c \sin^2 \tfrac{1}{2} A}{\sin^2 \tfrac{1}{2} (c \backsim b)}\right)^{\frac{1}{2}}$$

Assuming

2 $$\tan Z = \frac{\sin \tfrac{1}{2} A}{\sin \tfrac{1}{2} (c \backsim b)} (\sin b \sin c)^{\frac{1}{2}}$$

We have for the part under the radical:

$$(1 + \tan^2 Z)^{\frac{1}{2}} = \sec Z = \frac{1}{\cos Z}$$

And the formula becomes:

$$\sin \tfrac{1}{2} a = \frac{\sin \tfrac{1}{2}(c \backsim b)}{\cos Z} \qquad 3$$

2d Transformation. It is evident that it may also be reduced by a substitution analogous to Q, No. 4.

$$2 \sin b \sin c \sin^2 \tfrac{1}{2} A = 2 \cos^2 \tfrac{1}{2}(c \backsim b) - 2 \cos^2 \tfrac{1}{2} a$$

Whence may be deduced by the same process:

$$\cos \tfrac{1}{2} a = \cos \tfrac{1}{2}(c \backsim b) \left(1 - \frac{\sin b \sin c \sin^2 \tfrac{1}{2} A}{\cos^2 \tfrac{1}{2}(c \backsim b)}\right)^{\frac{1}{2}} \qquad 4$$

Assuming

$$\sin Z' = \frac{\sin \tfrac{1}{2} A}{\cos \tfrac{1}{2}(c \backsim b)} (\sin b \sin c)^{\frac{1}{2}} \qquad 5$$

The formula becomes:

$$\cos \tfrac{1}{2} a = \cos \tfrac{1}{2}(c \backsim b) \cos Z' \qquad 6$$

3d Transformation. Taking from problem 3, the formula for the value of $\cos^2 \tfrac{1}{2} A$, instead of the value of $\sin^2 \tfrac{1}{2} A$, we obtain:

$$2 \sin b \sin c \cos^2 \tfrac{1}{2} A = \cos a - \cos (b + c)$$

We have, consequently, the means to transform the equation again in two ways, by means of the two values of cosine a, and cosine $(b + c)$.

In the first place, by a substitution analogous to Q, No. 3, and the transformation made in No. 1:

$$\sin \tfrac{1}{2} a = \sin \tfrac{1}{2}(c + b) \left(1 - \frac{\sin b \sin c \cos^2 \tfrac{1}{2} A}{\sin^2 \tfrac{1}{2}(c + b)}\right)^{\frac{1}{2}} \qquad 7$$

In which, by assuming

$$\sin Z'' = \frac{\cos \tfrac{1}{2} A}{\sin \tfrac{1}{2}(c + b)} (\sin b \sin c)^{\frac{1}{2}} \qquad 8$$

The formula becomes :

9 $$\sin \tfrac{1}{2} a = \sin \tfrac{1}{2} (c + b) \cos Z''$$

4th Transformation. By a substitution analogous to that in the 2d transformation, we also obtain here :

$$2 \sin b \sin c \cos^2 \tfrac{1}{2} A = 2 \cos^2 \tfrac{1}{2} a - 2 \cos^2 \tfrac{1}{2} (c + b)$$

Whence in like manner :

$$\cos^2 \tfrac{1}{2} a = \cos^2 \tfrac{1}{2} (c + b) \left(1 + \frac{\sin b \sin c \cos^2 \tfrac{1}{2} A}{\cos^2 \tfrac{1}{2} (c + b)}\right)$$

10 $$\cos \tfrac{1}{2} a = \cos \tfrac{1}{2} (c + b) \left(1 + \frac{\sin b \sin c \cos^2 \tfrac{1}{2} A}{\cos^2 \tfrac{1}{2} (c + b)}\right)^{\frac{1}{2}}$$

Assuming again

11 $$\tan Z''' = \frac{\cos \tfrac{1}{2} A}{\cos \tfrac{1}{2} (c + b)} (\sin b \sin c)^{\frac{1}{2}}$$

the formula becomes :

12 $$\cos \tfrac{1}{2} a = \frac{\cos \tfrac{1}{2} (c + b)}{\cos Z'''}$$

The four preceding formulæ evidently furnish the means of forming two others for the tangents, as h, No. 3, and i, No. 3, in problems 3, and 4, in the following manner.

5th Transformation. Dividing No. 3, by No. 6 :

13 $$\frac{\sin \tfrac{1}{2} a}{\cos \tfrac{1}{2} a} = \tan \tfrac{1}{2} a = \frac{\sin \tfrac{1}{2} (c \backsim b)}{\cos \tfrac{1}{2} (c \backsim b) \cos Z \cos Z'} = \frac{\tan \tfrac{1}{2} (c \backsim b)}{\cos Z \cos Z'}$$

In which we have, as before, the auxiliary angles determined by No. 2 and 5.

6th Transformation. Dividing No. 9, above, by No. 12 :

$$\frac{\sin \tfrac{1}{2} a}{\cos \tfrac{1}{2} a} = \tan \tfrac{1}{2} a = \frac{\sin \tfrac{1}{2} (c + b) \cos Z'' \cos Z'''}{\cos \tfrac{1}{2} (c + b)}$$

14 $$= \tan \tfrac{1}{2} (c + b) \cos Z'' \cos Z'''$$

The values of cosine Z'', and cosine Z''', being determined by No. 8 and 11.

As these two formulæ have the same factors in the auxiliary arcs as the four preceding ones, and represent no more than two different functions, of which the same function is used finally, the calculation of them is not attended with much more labour; and this is fully compensated by their greater exactness and applicability to every case. The nature of the data must in this case, as in every other, determine the choice of the formula; if, for instance, $(c \backsim b)$ were small, the formulæ which employ its cosine would not afford exact results; it is, in general, better to employ those using $(c + b)$; and those giving the tangent, will in all cases be preferable. It will be readily perceived, that it is a matter of indifference whether we take the sine, or cosine, for the auxiliary arc, as well as, whether the tangent or cotangent is used, in the respective cases where their use occurs, provided the proper corresponding functions are also used in the final formula.

7th Transformation. We may also transform the original formula f, No. 3, of the preceding chapter. This gives a function of the whole angle, as follows; viz:

$$\cos a = \cos c \cos b + \sin b \sin c \cos A$$

by substituting for sine b, or sine c, the value of one or the other, in conformity with the principles whence we deduce series B.

In this manner, writing instead of sine b, its equal, tangent b cosine b, we have:

$$\cos a = \cos b \cos c + \tan b \cos b \sin c \cos A$$

And assuming: $\tan y = \cos A \tan b$, we obtain: 15

$$\cos a = \cos b \cos c + \cos b \sin c \tan y$$

Multiplying the two terms on the right hand by cosine y, we have:

$$\cos a = \frac{\cos b \cos c \cos y + \cos b \sin c \sin y}{\cos y}$$

And, as $\dfrac{\cos b}{\cos y}$ is a common factor:

$$\cos a = \frac{\cos b}{\cos y}(\cos c \cos y + \sin c \sin y)$$

16 $$= \frac{\cos b \,.\, \cos (c \backsim y)}{\cos y}$$

8th Transformation. Making, as before,

17 $$\cot y' = \cos A \tan b$$

we finally obtain:

18 $$\cos a = \frac{\cos b \sin (c + y')}{\sin y'}$$

It will be seen that these two formulæ are identical, for the passage from the tangent to the cosine is evidently the same as that from the cotangent to the sine. In the first case, we have the cosine of the difference between the auxiliary angle and the other side; and in the second case, the sine of the sum of the two angles; which produce again the identical trigonometric function.

§ 91. *Problem* 6. Given, two angles and the included side, to find the third angle.

B, *C*, and *a*, being given, to find *A*.

This case is exactly analogous to the preceding, as might, in fact, have been anticipated from the properties of the supplementary triangle; it, notwithstanding, requires a separate investigation, in consequence of the diversity that occurs in the algebraic signs, and in the combination of the simple arcs and their values; which occasions a change of sine into cosine. We therefore give these successive changes.

In preparing the formula k, No. 1, problem 4, we obtained:

$$2 \sin^2 \tfrac{1}{2} a = \frac{-\cos (B + C) - \cos A}{\sin B \sin C}$$

Whence

$$2 \sin B \sin C \sin^2 \tfrac{1}{2} a = -\cos (B + C) - \cos A$$

1st Transformation. Introducing the sines and cosines of the half angles instead of the cosines of the whole angles, by substitutions analogous to Q, No. 3 and 4:

$$2 \sin^2 \tfrac{1}{2} A = 2 \cos^2 \tfrac{1}{2} (B + C) + 2 \sin B \sin C \sin^2 \tfrac{1}{2} a$$

Dividing the whole by 2, making $\cos^2 \tfrac{1}{2} (B + C)$ a common factor, and extracting the root, the formula becomes:

$$\sin \tfrac{1}{2} A = \cos \tfrac{1}{2} (B + C) \left(1 + \frac{\sin B \sin C \sin^2 \tfrac{1}{2} a}{\cos^2 \tfrac{1}{2} (B + C)} \right)^{\frac{1}{2}} \qquad 1$$

Assuming again

$$\tan Z = \frac{\sin \tfrac{1}{2} a (\sin B \sin C)^{\frac{1}{2}}}{\cos \tfrac{1}{2} (B + C)} \qquad 2$$

The formula becomes:

$$\sin \tfrac{1}{2} A = \frac{\cos \tfrac{1}{2} (B + C)}{\cos Z} \qquad 3$$

2d Transformation. Substituting sine, and cosine of A, and $(B + C)$, in a manner the inverse of that employed above, we have, as may easily be seen:

$$\cos \tfrac{1}{2} A = \sin \tfrac{1}{2} (B + C) \left(1 - \frac{\sin B \sin C \sin^2 \tfrac{1}{2} a}{\sin^2 \tfrac{1}{2} (B + C)} \right)^{\frac{1}{2}} \qquad 4$$

Assuming

$$\sin Z' = \frac{\sin \tfrac{1}{2} a (\sin B \sin C)^{\frac{1}{2}}}{\sin \tfrac{1}{2} (B + C)} \qquad 5$$

The formula becomes:

$$\cos \tfrac{1}{2} A = \sin \tfrac{1}{2} (B + C) \cos Z'$$

3d Transformation. Taking from the 4th problem the preparation for the second transformation:

$$2 \cos^2 \tfrac{1}{2} a \sin B \sin C = \cos A + \cos (B \backsim C)$$

And substituting the functions of the half angles, exactly as in the preceding transformation, we have:

$$\sin^2 \tfrac{1}{2} A = \cos^2 \tfrac{1}{2} (B \backsim C) \left(1 - \frac{\sin B \sin C \cos^2 \tfrac{1}{2} a}{\cos^2 \tfrac{1}{2} (B \backsim C)}\right)$$

7 $$\sin \tfrac{1}{2} A = \cos \tfrac{1}{2} (B \backsim C) \left(1 - \frac{\sin B \sin C \cos^2 \tfrac{1}{2} a}{\cos^2 \tfrac{1}{2} (B \backsim C)}\right)^{\frac{1}{2}}$$

In which, assuming

8 $$\sin Z'' = \frac{\cos \tfrac{1}{2} a (\sin B \sin C)^{\frac{1}{2}}}{\cos \tfrac{1}{2} (B \backsim C)}$$

The formula becomes:

9 $$\sin \tfrac{1}{2} A = \cos \tfrac{1}{2} (B \backsim C) \cos Z''$$

4th Transformation. By substituting the functions of the half angles, in a manner the inverse of that used in the preceding transformation, we obtain, from the same formula as the preceding one, the following results in succession:

$$2 \cos^2 \tfrac{1}{2} a \sin B \sin C = 2 \cos^2 \tfrac{1}{2} A - 2 \sin^2 \tfrac{1}{2} (B \backsim C)$$

Whence

$$\cos^2 \tfrac{1}{2} A = \sin^2 \tfrac{1}{2} (B \backsim C) + \cos^2 \tfrac{1}{2} a \sin B \sin C$$

10 $$\cos \tfrac{1}{2} A = \sin \tfrac{1}{2} (B \backsim C) \left(1 + \frac{\sin B \sin C \cos^2 \tfrac{1}{2} a}{\sin^2 \tfrac{1}{2} (B \backsim C)}\right)^{\frac{1}{2}}$$

And assuming

11 $$\tan Z''' = \frac{\cos \tfrac{1}{2} a (\sin B \sin C)^{\frac{1}{2}}}{\sin \tfrac{1}{2} (B \backsim C)}$$

The formula becomes:

12 $$\cos \tfrac{1}{2} A = \frac{\sin \tfrac{1}{2} (B \backsim C)}{\cos Z'''}$$

5th Transformation. The four preceding transformations, also evidently give two for the tangents; viz.

By dividing formula 3, by formula 6, we obtain:

$$\frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A} = \tan \tfrac{1}{2} A = \frac{\cos \frac{1}{2}(B + C)}{\sin \frac{1}{2}(B + C) \cos Z \cos Z'} = \frac{\cot \frac{1}{2}(B + C)}{\cos Z \cos Z'} \qquad 13$$

for which Z, and Z', are determined by No. 2 and 8.

6th Transformation. By division of the formula No. 9, by the formula No. 12, is obtained:

$$\frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A} = \tan \tfrac{1}{2} A = \frac{\cos \frac{1}{2}(B \backsim C)}{\sin \frac{1}{2}(B \backsim C)} \cos Z'' \cos Z'''$$

$$= \cot \tfrac{1}{2}(B \backsim C) \cos Z'' \cos Z''' \qquad 14$$

where Z'', and Z''', are determined by No. 8 and 11, above.

7th Transformation. Taking the original formula, as in problem 4, we obtain, by transformations analogous to those for No. 16 and 18, in the 5th problem, the following results in succession, for two analogous transformations; viz:

$$\cos A = \cos a \sin B \sin C - \cos B \cos C$$

By a substitution, according to B, No. 8, using for sine B, its equal, tangent B cosine B:

$$\cos A = \cos a \tan B \cos B \sin C - \cos B \cos C$$

And assuming

$$\cot y = \cos a \tan B \qquad 15$$

$$\cos A = \cos B (\cot y \sin C - \cos C)$$

$$= \frac{\cos B}{\sin y} (\cos y \sin C - \cos C \sin y)$$

$$\cos A = \frac{\cos B}{\sin y} \sin (C - y) \qquad 16$$

. *8th Transformation.* This
and is thus obtained. If, instea
we assume :

17 $$\tan y' = c$$

we obtain :

$$\cos A = \cos B\,(\tan y'$$

$$= \frac{\cos B}{\cos y'}\,(\sin y'\ s$$

13 $$\cos A = \frac{-\cos B \cos (y'}{\cos y'}$$

It is necessary to pay strict atten
braic signs of $(y' + C)$ that deter
which is obtuse when $(y' + C)$ $\angle$
$(y' + C)$ ⦣ $90°$; and also to the alg
tions of y' itself, as determined by a

§ 92. *Problem 7.* Given, two sides
to find the remaining angles.

a, b, and C, being given, to find A,

By d, No. 4, we have :

$$\cot\left(\frac{C_{,} \backsim C_{,,}}{2}\right) = \frac{\cot \frac{1}{2}(B +}{\tan \frac{1}{2}(}$$

By d, No. 5, we have :

$$\cot\left(\frac{C_{,} \backsim C''}{2}\right) = \frac{\sin (a + b)}{\sin (a \backsim}$$

Therefore, by equality :

$$\frac{\cot \frac{1}{2}(A + B) \cot \frac{1}{2} C}{\tan \frac{1}{2}(A \backsim B)} = \frac{\sin (a + b)}{\sin (a \backsim}$$

Which gives the two following equations :

tely omitting the accentuation, we have as a final

o

$$n \tfrac{1}{2} (a + b) = \tan \tfrac{1}{2} c \frac{\cos \tfrac{1}{2} (A \backsim B)}{\cos \tfrac{1}{2} (A + B)} \quad 1$$

$$\tfrac{1}{2} (a \backsim b) = \tan \tfrac{1}{2} c \frac{\sin \tfrac{1}{2} (A \backsim B)}{\sin \tfrac{1}{2} (A + B)} \quad 2$$

here, are evidently found as the two angles
ner instance.

$+ \tfrac{1}{2} (a \backsim b)$; $\quad b = \tfrac{1}{2} (a + b) - \tfrac{1}{2} (a \backsim b)$

ith the two angles opposite to them are given,
nd o, determine with equal ease the third
rd side.

). Given, two sides and an angle opposite
find the third side.

g given, to find c.

have:

$\cos c \cos b + \sin b \sin c \cos A$

lready been transformed, in problem 5,
18, into the two following.

Making

p

$y = \tan b \cos A$ 1

$(c \backsim y)$

the present problem:

2

2 $$\tan \tfrac{1}{2}(A \backsim B) = \cot \tfrac{1}{2} C \frac{\sin \frac{1}{2}(a \backsim b)}{\sin \frac{1}{2}(a+b)}$$

These formulæ, which are easy to calculate, and advantageous, enable us to dispense with the research of others, that might be easily constructed, in which no more than a single angle is determined. They give, of course, directly:

$$A = \tfrac{1}{2}(A+B) + \tfrac{1}{2}(A \backsim B)$$

and $$B = \tfrac{1}{2}(A+B) - \tfrac{1}{2}(A \backsim B)$$

§ 93. *Problem* 8. Given, two angles and the contained side, to find the two remaining sides.

A, B, and *c*, being given, to find *a*, and *b*.

The consideration of the formulæ of the same series whence the preceding have been derived, shows that formulæ similar to them may be obtained for this case. But in order to shorten the operation, we shall here proceed by means of the supplementary triangle; writing each of the angles and sides in expressions taken from the supplementary triangle; distinguishing them by accentuation until reduced. When reduced by this method, the formulæ n, No. 1 and 2, become:

$$\tan \frac{180° - a' + 180° - b'}{2}$$
$$= \cot(90° - \tfrac{1}{2}c') \frac{\cos \frac{1}{2}\left((180° - A') \backsim (180° - B')\right)}{\cos \frac{1}{2}\left((180° - A') + (180° - B')\right)}$$

$$\tan \tfrac{1}{2}\left((180° - a') \backsim (180° - b')\right)$$
$$= \cot(90° - \tfrac{1}{2}c') \frac{\sin \frac{1}{2}\left((180° - A') \backsim (180° - B')\right)}{\sin \frac{1}{2}\left((180° - A') + (180° - B')\right)}$$

Making the compensations which evidently result throughout, considering that $\cos(180° - x) = -\cos x$; that $\cot(90° - \frac{1}{2}c') = \tan \frac{1}{2}c'$, and that $\tan(180° - x) = -\tan x$,

and ultimately omitting the accentuation, we have as a final result:

o

$$\tan \tfrac{1}{2}(a + b) = \tan \tfrac{1}{2} c \, \frac{\cos \tfrac{1}{2}(A \backsim B)}{\cos \tfrac{1}{2}(A + B)} \qquad 1$$

$$\tan \tfrac{1}{2}(a \backsim b) = \tan \tfrac{1}{2} c \, \frac{\sin \tfrac{1}{2}(A \backsim B)}{\sin \tfrac{1}{2}(A + B)} \qquad 2$$

The two sides here, are evidently found as the two angles were in the former instance.

$$a = \tfrac{1}{2}(a + b) + \tfrac{1}{2}(a \backsim b)\,; \qquad b = \tfrac{1}{2}(a + b) - \tfrac{1}{2}(a \backsim b)$$

If two sides with the two angles opposite to them are given, the formulæ n, and o, determine with equal ease the third angle and the third side.

§ 94. ***Problem* 9. Given, two sides and an angle opposite to one of them, to find the third side.**

b, a, and A, being given, to find c.

By f, No. 3, we have:

$$\cos a = \cos c \cos b + \sin b \sin c \cos A$$

This formula has already been transformed, in problem 5, formulæ l, No. 15 to 18, into the two following.

1*st Transformation.* Making

p

$$\tan y = \tan b \cos A \qquad 1$$

we there obtained:

$$\cos a = \frac{\cos b}{\cos y} \cos (c \backsim y)$$

Whence, dividing by $\dfrac{\cos b}{\cos y}$, we obtain for the present problem:

$$\cos (c \backsim y) = \frac{\cos a \cos y}{\cos b} \qquad 2$$

2d Transformation. Making

3 $$\cot y' = \cos A \tan b$$

we there obtained:

$$\cos a = \frac{\cos b}{\sin y'} \sin (c + y')$$

Whence we deduce, as before:

4 $$\sin (c + y') = \frac{\cos a \sin y'}{\cos b}$$

These two formulæ give, each, one of the two possible angles. Only one, however, need be calculated, because this double result is also obtained by taking both the sum, and the difference between the two angles, c, and $c \pm y$; with this understanding, therefore, the two formulæ are identical.

3d Transformation. By a direct analytical treatment of the formula f, No. 3, we may obtain a mutation giving the side c, in a formula analogous to F, No. 1, or 2; that is to say, in two parts; the sum or difference of which will be the side c, sought; this process leads through a quadratic equation, which may be avoided by proceeding as has been done in Plane Trigonometry, *problem* 5.

Making, therefore, according to *figure* 15, *or* 16:

5 $$c = BD \pm AD = x \pm y$$

We have by series b, No. 3:

6 $$\tan AD = \tan b \cos A = \tan x$$

7 and $$\tan BD = \tan a \cos B = \tan y$$

The angle B, is not given directly; but is determinable from the data of the problem; we have by them:

8 $$\sin B = \frac{\sin A \sin b}{\sin a}$$

This angle, B, therefore, forms an auxiliary arc in the determination of tangent y, by which means both parts of c are, therefore, determined by their tangents.

4th Transformation. If we express, by means of the value of the tangents obtained in the preceding, the values of the cosines of x and y, according to the formula D, No. 9, we obtain the following simple expressions, which are extremely easy to calculate. We have, in that case:

$$\cos x = \frac{1}{(1 + \tan^2 x)^{\frac{1}{2}}}$$

and in this, by 6:

$$\tan x = \frac{\sin b \cos A}{\cos b}$$

Thence:

$$\cos x = \frac{1}{\left(1 + \frac{\sin^2 b \cos A}{\cos^2 b}\right)^{\frac{1}{2}}}$$

$$= \frac{\cos b}{(\cos^2 b + \sin^2 b \cos^2 A)^{\frac{1}{2}}}$$

$$= \frac{\cos b}{(1 - \sin^2 b \sin^2 A)^{\frac{1}{2}}} \quad 9$$

And by changing the expression for the value of tangent y, in like manner, we obtain: by first substituting in 7, the value of $\cos B = (1 - \sin^2 B)^{\frac{1}{2}}$

$$\tan y = \frac{\sin a}{\cos a}\left(1 - \frac{\sin^2 A \sin^2 b}{\sin^2 a}\right)^{\frac{1}{2}}$$

Substituting this value in the same formula, D, No. 9, we obtain:

$$\cos y = \frac{1}{(1 + \tan^2 y)^{\frac{1}{2}}}$$

$$= \frac{1}{\left(1 + \frac{\sin^2 a}{\cos^2 a}\left(1 - \frac{\sin^2 A \sin^2 b}{\sin^2 a}\right)\right)^{\frac{1}{2}}}$$

$$= \frac{1}{\left(1 + \frac{1}{\cos^2 a}(\sin^2 a - \sin^2 A \sin^2 b)\right)^{\frac{1}{2}}}$$

$$= \frac{\cos a}{(\cos^2 a + \sin^2 a - \sin^2 A \sin^2 b)^{\frac{1}{2}}}$$

10 $$\cos y = \frac{\cos a}{(1 - \sin^2 A \sin^2 b)^{\frac{1}{2}}}$$

These two formulæ have the same denominator; they furnish, consequently, the same auxiliary arc for both; making, therefore:

11 $$\sin Z = \sin A \sin b$$

we finally obtain, for the two formulæ 9 and 10, the definitive values:

12 $$\cos x = \frac{\cos b}{\cos Z}$$

and

13 $$\cos y = \frac{\cos a}{\cos Z}$$

the calculation of which is evidently of the greatest simplicity.

By using D, No. 3, instead of No. 9, somewhat similar expressions are obtained for the sines; but, as they are not so simple as either of the preceding ones, they are not here deduced.

§ 95. ***Problem* 10. Given, two angles and a side opposite to one of them; to find the third angle.**

Given, A, B, a; to find C.

The general formula for this case is again:

$$\cos C = \sin B \sin A \cos c - \cos B \cos A$$

1*st Transformation.* It is evident that, making use of the 7*th transformation of problem* 6, we obtain by simple division of m, No. 16:

Assuming q

$$\cot y = \cos a \tan B \qquad 1$$

$$\sin (C \backsim y) = \frac{\cos A \sin y}{\cos B} \qquad 2$$

2*d Transformation.* By the same process, we obtain from m, No. 18, upon the supposition that:

$$\tan y' = \cos a \tan B \qquad 3$$

$$\cos (C + y') = - \frac{\cos A \cos y'}{\cos B} \qquad 4$$

These two formulæ are evidently under the same predicament as the two corresponding ones in the preceding problem.

3*d Transformation.* To this, what has been said in the 3*d transformation of problem* 9, again applies exactly; for we have here again, by *figure* 15 *and* 16:

$$C = BCD \pm ACD = x \pm y \qquad 5$$

By series b, No. 5, we obtain for this case:

$$\cot x = \cos a \tan B \qquad 6$$

$$\cot y = \cos b \tan A \qquad 7$$

The unknown side b, employed here, is determined according to *problem* 1, as an auxiliary arc for No. 7, thus:

$$\sin b = \frac{\sin a \sin B}{\sin A}$$

4th Transformation. The results of the preceding formula being used, in the same manner as in the 4th transformation of the preceding problem, but applied to D, No. 2, will give for this problem the following transformation, exactly analogous in point of form. We have there:

$$\sin x = \frac{1}{(1 + \cot^2 x)^{\frac{1}{2}}}$$

From No. 6, above:

$$\cot x = \cos a \frac{\sin B}{\cos B}$$

Whence

$$\sin x = \frac{1}{\left(1 + \frac{\cos^2 a \sin^2 B}{\cos^2 B}\right)^{\frac{1}{2}}}$$

$$= \frac{\cos B}{(\cos^2 B + \sin^2 B \cos^2 a)^{\frac{1}{2}}}$$

$$= \frac{\cos B}{(\cos^2 B + \sin^2 B - \sin^2 B \sin^2 a)^{\frac{1}{2}}}$$

9 $$\sin x = \frac{\cos B}{(1 - \sin^2 B \sin^2 a)^{\frac{1}{2}}}$$

And in like manner, after having inserted the auxiliary angle No. 8, we have:

$$\cot y = \frac{\sin A}{\cos A}\left(1 - \frac{\sin^2 a \sin^2 B}{\sin^2 A}\right)^{\frac{1}{2}}$$

$$= \frac{(\sin^2 A - \sin^2 a \sin^2 B)^{\frac{1}{2}}}{\cos A}$$

Whence is obtained, by analogy to D, No. 2:

$$\sin y = \frac{1}{\left(1 + \frac{(\sin^2 A - \sin^2 a \sin^2 B)}{\cos^2 A}\right)^{\frac{1}{2}}}$$

$$= \frac{\cos A}{(\cos^2 A + \sin^2 A - \sin^2 a \sin^2 B)^{\frac{1}{2}}}$$

$$\sin y = \frac{\cos A}{(1 - \sin^2 a \sin^2 B)^{\frac{1}{2}}} \qquad 10$$

Here, again, the denominators are equal; and the auxiliary arc

$$\sin Z = \sin a \sin B \qquad 11$$

Which gives the final formulæ :

$$\sin x = \frac{\cos B}{\cos Z} \qquad 12$$

and

$$\sin y = \frac{\cos A}{\cos Z} \qquad 13$$

§ 96. *Problem* 11. Given, two angles, and a side opposite to one of them; to find the side contained between these angles.

Given, B, a, A; to find c.

The solution of this case depends upon the original formula, f, No. 2.

$$\sin c \cot a + \cos c \cos B = \sin B \cot A$$

1*st Transformation.* By making

$$\tan x = \cos B \tan a \qquad 1$$

which gives:

$$\cot a = \frac{\cos B \cos x}{\sin x}$$

T

And substituting it in the equation, it becomes:

$$\frac{\sin c \cos B \cos x}{\sin x} + \cos c \cos B = \sin B \cot A$$

Reducing to a common denominator, and making $\dfrac{\cos B}{\sin x}$ a common factor:

$$\frac{\cos B}{\sin x}(\sin c \cos x + \cos c \sin x) = \sin B \cot A$$

$$\frac{\cos B \sin (c + x)}{\sin x} = \sin B \cot A$$

And finally:

2 $$\sin (c + x) = \tan B \cot A \sin x$$

2d Transformation. This is obtained in a manner similar to the foregoing, by making:

3 $$\cot x' = \cos B \tan a$$

Which gives:

$$\cot a = \frac{\cos B \sin x'}{\cos x'}$$

Which, being substituted in the formula, gives:

$$\frac{\sin c \cos B \sin x'}{\cos x'} + \cos c \cos B = \sin B \cot A$$

Treating this as in the previous case, we obtain successively:

$$\frac{\cos B}{\cos x'}(\sin c \sin x' + \cos c \cos x') = \sin B \cot A$$

$$\frac{\cos B \cos (c \backsim x')}{\cos x'} = \sin B \cot A$$

4 $$\cos (c \backsim x') = \tan B \cot A \cos x'$$

that are two formulæ, in the same predicament as q, No. 2 and 4.

3*d* *Transformation.* This is made in a manner analogous to that in *problem* 9, and equally avoids the quadratic equation, by which it could be obtained from f, No. 3, or e, No. 3; taking, according to *figure* 15 *and* 16, the segments of the side:

$$c = BD \pm AD = x \pm y \qquad 5$$

And taking, in b, No. 4, the value of these parts, we obtain:

$$\tan x = \tan a \cos B \qquad 6$$

$$\tan y = \tan b \cos A \qquad 7$$

The side *b*, which is here supposed to be given, and must therefore be determined as an auxiliary angle from the data of the problem, is:

$$\sin b = \frac{\sin B \sin a}{\sin A} \qquad 8$$

By which, again, all parts are solved, and the result calculable by logarithms.

4*th* *Transformation.* From this last formula can be deduced another, in the same way as in the two preceding problems, for the same two sections x, and y, of the side c; thus:

Expressing the cosine by D, No. 9, we have:

$$\cos x = \frac{1}{(1 + \tan^2 x)^{\frac{1}{2}}}$$

Which gives here, by substituting the value of tangent x, that we have just obtained:

$$\cos x = \frac{1}{(1 + \tan^2 a \cos^2 B)^{\frac{1}{2}}}$$

$$= \frac{\cos a}{(\cos^2 a + \cos^2 B \sin^2 a)^{\frac{1}{2}}}$$

9 $$\cos x = \frac{\cos a}{(1 - \sin^2 a \sin^2 B)^{\frac{1}{2}}}$$

And taking D, No. 3, to express sine y in terms of tangent y, we have:

$$\sin y = \frac{\tan y}{(1 + \tan^2 y)^{\frac{1}{2}}}$$

And expressing tangent b by the auxiliary arc, we have:

$$\tan b = \frac{\sin a \sin B}{\sin A \left(1 - \dfrac{\sin^2 a \sin^2 B}{\sin^2 A}\right)^{\frac{1}{2}}}$$

$$= \frac{\sin a \sin B}{(\sin^2 A - \sin^2 a \sin^2 B)^{\frac{1}{2}}}$$

Whence

$$\tan y = \frac{\cos A \sin a \sin B}{(\sin^2 A - \sin^2 a \sin^2 B)^{\frac{1}{2}}}$$

and

$$\sin y = \frac{\dfrac{\cos A \sin a \sin B}{(\sin^2 A - \sin^2 a \sin^2 B)^{\frac{1}{2}}}}{\left(1 + \dfrac{\sin^2 a \sin^2 B \cos^2 A}{\sin^2 A - \sin^2 a \sin^2 B}\right)^{\frac{1}{2}}}$$

Bringing the denominator to a common denominator, and compensating, in numerator and denominator:

$$\sin y = \frac{\cos A \sin a \sin B}{(\sin^2 A - \sin^2 a \sin^2 B + \sin^2 a \sin^2 B \cos^2 A)^{\frac{1}{2}}}$$

$$= \frac{\cos A \sin a \sin B}{\left(\sin^2 A - \sin^2 a \sin^2 B\,(1 - \cos^2 A)\right)^{\frac{1}{2}}}$$

$$\sin y = \frac{\cos A \sin a \sin B}{\sin A\,(1 - \sin^2 a \sin^2 B)^{\frac{1}{2}}}$$

$$= \frac{\cot A \sin a \sin B}{(1 - \sin^2 a \sin^2 B)^{\frac{1}{2}}}$$ 10

Here we have again, for the determination of x and y, the same denominator, and therefore the same auxiliary arc. Therefore, making

$$\sin Z = \sin a \sin B$$ 11

we obtain finally the two following formulæ for calculation:

$$\cos x = \frac{\cos a}{\cos Z}$$ 12

and

$$\sin y = \frac{\cot A \sin a \sin B}{\cos Z} = \cot A \tan Z$$ 13

Of these two expressions for sine y, the first will be found shorter in the actual calculation, because it is easier to write sine a sine B, twice, and use the same auxiliary arc, than to take two different auxiliary angles.

§ 97. *Problem* 12. Given, two sides and an angle opposite to one of them; to find the contained angle.

Given, A, b, a; to find C.

By f, No. 2, we have:

s

$$\sin C \cot A + \cos b \cos C = \sin b \cot a$$ 1

1*st Transformation* is obtained as in the preceding problem, by making

$$\tan x = \cos b \tan A$$

Which gives:

$$\cot A = \frac{\cos b \cos x}{\sin x}$$

The equation becomes, by this substitution:

$$\frac{\cos C \cos b \cos x}{\sin x} + \cos b \cos C = \sin b \cot a$$

Reducing to a common denominator, and making $\dfrac{\cos b}{\sin x}$ a common factor, gives:

$$\frac{\cos b}{\sin x} (\sin C \cos x + \cos C \sin x) = \sin b \cot a$$

or

$$\frac{\cos b \sin (C + x)}{\sin x} = \sin b \cot a$$

Whence, finally:

2 $$\sin (C + x) = \cot a \tan b \sin x$$

2d Transformation. A formula corresponding to the preceding is obtained, by making

3 $$\cot x' = \cos b \tan A$$

And following the same process in the reduction of this as in the preceding, the final formula will become:

4 $$\cos (C \backsim x') = \tan b \cot a \cos x'$$

These two formulæ are again to be considered and treated as the two, q, No. 2 and 4.

3d Transformation. Here, as in problem 10, expressing the two segments of the angle C, we obtain by means of b, No. 5, for each of them, simple formulæ for logarithmic calculation; thus:

5 $$C = BCD \pm ACD = x \pm y$$

We have by b, No. 5:

6 $$\cot x = \cos a \tan B$$

and

7 $$\cot y = \cos b \tan A$$

Where B is to be determined from the data of the problem, and used as an auxiliary angle; thus:

$$\sin B = \frac{\sin b \sin A}{\sin a}. \qquad 8$$

4th Transformation. We may here again apply with advantage, the transformations given in formulæ 6, and 7, by the aid of D, No. 14 and 21.

We have from 8, the value:

$$\tan B = \frac{\sin b \sin A}{\sin a \left(1 - \frac{\sin^2 b \sin^2 A}{\sin^2 a}\right)^{\frac{1}{2}}}$$

$$= \frac{\sin b \sin A}{(\sin^2 a - \sin^2 b \sin^2 A)^{\frac{1}{2}}}$$

Whence

$$\cot x = \cos a \tan B = \frac{\cos a \sin b \sin A}{(\sin^2 a - \sin^2 b \sin^2 A)^{\frac{1}{2}}}$$

By D, No. 14, we have:

$$\cos x = \frac{\cot x}{(1 + \cot^2 x)^{\frac{1}{2}}} = \frac{\frac{\cos a \sin b \sin A}{(\sin^2 a - \sin^2 b \sin^2 A)^{\frac{1}{2}}}}{\left(1 + \frac{\sin^2 b \cos^2 a \sin^2 A}{\sin^2 a - \sin^2 b \sin^2 A}\right)^{\frac{1}{2}}}$$

$$= \frac{\cos a \sin b \sin A}{(\sin^2 a - \sin^2 b \sin^2 A + \sin^2 b \cos^2 a \sin^2 A)^{\frac{1}{2}}}$$

$$= \frac{\cos a \sin b \sin A}{\left(\sin^2 a - \sin^2 b \sin^2 A\,(1 - \cos^2 a)\right)^{\frac{1}{2}}}$$

$$= \frac{\cos a \sin b \sin A}{\sin a\,(1 - \sin^2 b \sin^2 A)^{\frac{1}{2}}}$$

9 $$\cos x = \frac{\cot a \sin b \sin A}{(1 - \sin^2 b \sin^2 A)^{\frac{1}{2}}}$$

From D, No. 2, we have for

$$\sin y = \frac{1}{(1 + \cot^2 y)^{\frac{1}{2}}} = \frac{1}{\left(1 + \frac{\cos^2 b \sin^2 A}{\cos^2 A}\right)^{\frac{1}{2}}}$$

$$= \frac{\cos A}{(\cos^2 A + \cos^2 b \sin^2 A)^{\frac{1}{2}}}$$

$$= \frac{\cos A}{(\cos^2 A + \sin^2 A - \sin^2 A \sin^2 b)^{\frac{1}{2}}}$$

10 $$\sin y = \frac{\cos A}{(1 - \sin^2 A \sin^2 b)^{\frac{1}{2}}}$$

The two formulæ 9 and 10, thus obtained, have again the property of having the same denominator; therefore, making use of the same auxiliary angle, namely:

11 $$\sin Z = \sin A \sin b$$

the final formulæ for calculation become:

12 $$\cos x = \frac{\cot a \sin b \sin A}{\cos Z} = \cot a \tan Z$$

and

13 $$\sin y = \frac{\cos A}{\cos Z}$$

What has been said in relation to r, No. 13, also applies here to No. 12.

§ 98. We have thus obtained for each of the cases of oblique angled spherical trigonometry, a variety of formulæ, of easy calculation by logarithms; as it may be useful in prac-

tice to have complete formulæ for every case, we have considered it proper to enter into these details in this part of the treatise, in order that a choice may be made among the formulæ of such as may best suit in any individual case, and afford the greatest accuracy. A skilful calculator will also find in them a check upon his own numeric operations; for he may at the same time calculate by means of two different formulæ; for which purpose he will choose such as are most easily used simultaneously, in consequence of their only differing from each other in the employment of different trigonometric functions of the same elements.

These formulæ all concur in showing: that, in Spherical Trigonometry, under equal circumstances, the different parts equally depend upon their data for their form of combination; the part sought may be either a side or an angle; so that there are in truth only six forms of this mutual dependence of the parts, which differ only in the details of signs, and occasional changes between sines and cosines, or tangents and cotangents.

It may be easily conceived, when we consider the multitude of analytic formulæ that may be deduced from the nature of the trigonometric functions: that other formulæ and transformations, besides those here presented, are possible, as well as other methods; but those here given are, in general, the most direct, and most accurate, and are consequently of most frequent use.

In all the above transformations, the formulæ from which they originate, or any particular operation performed, which may not be immediately evident, has been quoted and referred to, and, in addition, the aim of any operation, and the intended mode, has generally been quoted before the operation; but it has been uniformly supposed, as stated in the beginning, that series B and C were known, as it is supposed that any arithmetician knows his multiplication table; though it is not necessary to learn them by heart, for the proper study

of the first elements, and practice, will very soon make them as familiar as the multiplication table is to calculators.

§ 99. In order to decide the doubtful cases, as indicated by the formulæ, or the nature of the cases, we may observe a few general rules.

1. That in the formulæ 1, 2, 3, and 4, of the problems 9, 10, 11, and 12, the tangent or cotangent of the auxiliary arc, and the cosines of the other parts, may change sign; which, therefore, must be attended to.

2. That by never employing a triangle with a side or angle exceeding 80°, the results that would lead to such a side or angle are of course excluded.

3. That in every triangle, the greatest sides and greatest angles are opposite, the least side to the least angle, and the mean to the mean.

4. The principle, that the sum of the three sides of the triangle is always less, and the sum of the three angles, always more than four right angles, sometimes gives another criterion to judge in the case; as well as: that the sum of the angles shall not exceed six right angles.

5. The circumstances of a given case rarely leave room for doubt in the decision. It has already been observed: that wherever the case is not doubtful by nature, the formulæ giving half angles are the most advantageous, in this, as well as in other respects.

TABLE of the FORMULÆ of Oblique Angled Plane Trigonometry.

Series.	No.	Data.	Sought.	Analytical Formulæ for Calculation.	No. (Auxiliary Arcs or Elements)	Auxiliary.	Analytic Formulæ.
Y	1	b, c, B	C	$\sin C = \dfrac{c \sin B}{b}$			
		c, B, C	c	$c = \dfrac{b \sin C}{\sin B}$			
	2	b, c, A	$B - C$	$\tan \frac{1}{2}(B - C) = \cot \frac{1}{2} A \dfrac{b - c}{b + c}$			
				$B = 90° - \frac{1}{2} A + \frac{1}{2}(B - C)$			
				$C = 90° - \frac{1}{2} A - \frac{1}{2}(B - C)$			
	4			$\tan \frac{1}{2}(B - C) = \cot \frac{1}{2} A \tan (45° - Z)$	3	$\tan Z$	$= \dfrac{c}{b}$
	8	a, B, c	b	$b = \dfrac{a \backsim c}{\cos x}$	7	$\tan x$	$= \dfrac{2 \sin \frac{1}{2} B}{a \backsim c} (a\, c)^{\frac{1}{2}}$
				$b = (c + a) \sin x$	10	$\cos x$	$= \dfrac{2 \cos \frac{1}{2} B}{a + c} (a\, c)^{\frac{1}{2}}$

Series.	No.	Data.	Sought.	Analytical Formulæ for Calculation.	No.	Auxiliary.	Analytic Formulæ.
Y	15	a, b, c	B	$\sin \frac{1}{2} B = \left(\frac{(p-c)(p-a)}{a\,c}\right)^{\frac{1}{2}}$	14	p	$= \frac{a+b+c}{2}$
	16			$\cos \frac{1}{2} B = \left(\frac{p\,(p-b)}{a\,c}\right)^{\frac{1}{2}}$			
	17			$\tan \frac{1}{2} B = \left(\frac{(p-a)(p-c)}{p\,(p-b)}\right)^{\frac{1}{2}}$		$\sin B$	$= \frac{b \sin C}{c}$
	20	b, C, c	a	$a = \frac{b \cos C}{\cos^2 y}$ or $a = b \cos C \cos^2 y$	19	$\left.\begin{matrix}\tan\\ \text{or}\\ \sin\end{matrix}\right\} y$	$= \left(\frac{c \,.\, \cos B}{b \,.\, \cos C}\right)^{\frac{1}{2}}$

N. B.—The formulæ of Right Angled Plane Trigonometry appear already in a tabular form in series A; and need not therefore be repeated here.

FORMULÆ for the Surface of Triangles.

Series.	No.	Data.	Sought.	Analytical Formulæ for Calculation.	AUXILIARY ARCS OR ELEMENTS. Auxiliary.	Analytic Formulæ.
Z	1 and 2	[illegible]	S	$S = \dfrac{a^2 \sin b \sin C}{2 \sin (B + C)} = \dfrac{a^2}{2 (\cot B + \cot C)}$		
	3	a, C, b	S	$S = \dfrac{a\, b \sin C}{2}$		
	5	a, b, c	S	$S = (p\,(p-a)\,(p-b)\,(p-c))^{\frac{1}{2}}$	p	$= \dfrac{a+b+c}{2}$
	6	b, C, c	S	$S = \dfrac{b^2 \cos C \sin C}{2} \pm \dfrac{c\,.\,b \sin C}{2} \left(1 - \dfrac{b^2 \sin^2 C}{c^2}\right)^{\frac{1}{2}}$		

TABLE of the FORMULÆ of Oblique Angled Spherical Trigonometry.

Data.	Sought.	Series.	No.	Function determined.		Analytical Formulæ for the part sought.	Auxiliary Arcs or Elements. No.	Auxiliary.		Formulæ for the Auxiliary.
c, B, b	C	g		$\sin C$	=	$\dfrac{\sin B \sin c}{\sin b}$				
B, c, C	b	h		$\sin b$	=	$\dfrac{\sin B \sin c}{\sin C}$				
a, b, c	A	i	1	$\sin \frac{1}{2} A$	=	$\left(\dfrac{\sin (p - c) \sin (p - b)}{\sin c \sin b}\right)^{\frac{1}{2}}$		p	=	$\dfrac{a + b + c}{2}$
			2	$\cos \frac{1}{2} A$	=	$\left(\dfrac{\sin p \sin (p - a)}{\sin b \sin c}\right)^{\frac{1}{2}}$				
			3	$\tan \frac{1}{2} A$	=	$\left(\dfrac{\sin (p - c) \sin (p - b)}{\sin p \sin (p - a)}\right)^{\frac{1}{2}}$				
A, B, C	a	k	1	$\sin \frac{1}{2} a$	=	$\left(\dfrac{\cos P \cos (P - A)}{\sin B \sin C}\right)^{\frac{1}{2}}$		P	=	$\dfrac{A + B + C}{2}$
			2	$\cos \frac{1}{2} a$	=	$\left(\dfrac{\cos (P - C) \cos (P - B)}{\sin B \sin C}\right)^{\frac{1}{2}}$				

TABLE of the FORMULÆ of Oblique Angled Spherical Trigonometry, Continued.

Data.	Sought.	Series.	No.	Function determined.	Anlyatical Formulæ for the part sought.	AUXILIARY ARCS OR ELEMENTS. No.	Auxiliary.	Formulæ for the Auxiliary.
A, B, C	a	k	3	$\tan \frac{1}{2} a$	$= \left(\dfrac{\cos P \cos (P - A)}{\cos (P - C) \cos (P - B)}\right)^{\frac{1}{2}}$			
A, c, b	a	1	3	$\sin \frac{1}{2} a$	$= \dfrac{\sin \frac{1}{2} (c \backsim b)}{\cos Z}$	2	$\tan Z$	$= \dfrac{\sin \frac{1}{2} A}{\sin \frac{1}{2} (c \backsim b)} (\sin b \sin c)^{\frac{1}{2}}$
			6	$\cos \frac{1}{2} a$	$= \cos \frac{1}{2} (c \backsim b) \cos Z'$	5	$\sin Z'$	$= \dfrac{\sin \frac{1}{2} A}{\cos \frac{1}{2} (c \backsim b)} (\sin b \sin c)^{\frac{1}{2}}$
			9	$\sin \frac{1}{2} a$	$= \sin \frac{1}{2} (c + b) \cos Z''$	8	$\sin Z''$	$= \dfrac{\cos \frac{1}{2} A}{\sin \frac{1}{2} (c + b)} (\sin b \sin c)^{\frac{1}{2}}$
			12	$\cos \frac{1}{2} a$	$= \dfrac{\cos \frac{1}{2} (c + b)}{\cos Z'''}$	11	$\tan Z'''$	$= \dfrac{\cos \frac{1}{2} A}{\cos \frac{1}{2} (c + b)} (\sin b \sin c)^{\frac{1}{2}}$
			13	$\tan \frac{1}{2} a$	$= \dfrac{\tan \frac{1}{2} (c \backsim b)}{\cos Z \cos Z'}$			
			14	$\tan \frac{1}{2} a$	$= \tan \frac{1}{2} (c + b) \cos Z'' \cos Z'''$			
			16	$\cos a$	$= \dfrac{\cos b \cos (c \backsim y)}{\cos y}$	15	$\tan y$	$= \cos A \tan b$

Data.	Sought.	Series.	No.	Function determined.	Analytical Formulæ for the part sought.	No.	Auxiliary.	Formulæ for the Auxiliary.
A, c, b	a	1	18	$\cos a$	$= \dfrac{\cos b \sin (c + y')}{\sin y'}$	17	$\cot y'$	$= \cos A \tan b$
B, a, C	A	m	3	$\sin \frac{1}{2} A$	$= \dfrac{\cos \frac{1}{2} (B + C)}{\cos Z}$	2	$\tan Z$	$= \dfrac{\sin \frac{1}{2} a}{\cos \frac{1}{2} (B + C)} (\sin B \sin C)^{\frac{1}{2}}$
			6	$\cos \frac{1}{2} A$	$= \sin \frac{1}{2} (B + C) \cos Z'$	5	$\sin Z'$	$= \dfrac{\sin \frac{1}{2} a}{\sin \frac{1}{2} (B + C)} (\sin B \sin C)^{\frac{1}{2}}$
			9	$\sin \frac{1}{2} A$	$= \cos \frac{1}{2} (B \backsim C) \cos Z''$	8	$\sin Z''$	$= \dfrac{\cos \frac{1}{2} a}{\cos \frac{1}{2} (B \backsim C)} (\sin B \sin C)^{\frac{1}{2}}$
			12	$\cos \frac{1}{2} A$	$= \dfrac{\sin \frac{1}{2} (B \backsim C)}{\cos Z'''}$	11	$\tan Z'''$	$= \dfrac{\cos \frac{1}{2} a}{\sin \frac{1}{2} (B \backsim C)} (\sin B \sin C)^{\frac{1}{2}}$
			13	$\tan \frac{1}{2} A$	$= \dfrac{\cot \frac{1}{2} (B + C)}{\cos Z \cos Z'}$			
			14	$\tan \frac{1}{2} A$	$= \cot \frac{1}{2} (B \backsim C) \cos Z'' \cos Z'''$			
			16	$\cos A$	$= \dfrac{\cos B \sin (C - y)}{\sin y}$	$\cancel{15}$	$\cancel{\cot y}$	$\cancel{= \cos a \tan B}$

TABLE of the FORMULÆ of Oblique Angled Spherical Trigonometry, Continued.

Data.	Sought.	Series.	No.	Function determined.	Analytical Formulæ for the part sought.	Auxiliary Arcs or Elements. No.	Auxiliary.	Formulæ for the Auxiliary.
B, a, C	A	m	18	$\cos A$	$= \dfrac{-\sin B \cos (C + y')}{\cos y'}$	17	$\tan y'$	$= \cos a \tan B$
C, a, b	A, & B	n	1	$\tan \frac{1}{2} (A + B)$	$= \cot \frac{1}{2} C \dfrac{\cos \frac{1}{2} (a \backsim b)}{\cos \frac{1}{2} (a + b)}$			
			2	$\tan \frac{1}{2} (A \backsim B)$	$= \cot \frac{1}{2} C \dfrac{\sin \frac{1}{2} (a \backsim b)}{\sin \frac{1}{2} (a + b)}$			
A, c, B	a, & b	o	1	$\tan \frac{1}{2} (a + b)$	$= \tan \frac{1}{2} c \dfrac{\cos \frac{1}{2} (B \backsim A)}{\cos \frac{1}{2} (A + B)}$			
			2	$\tan \frac{1}{2} (a \backsim b)$	$= \tan \frac{1}{2} c \dfrac{\sin \frac{1}{2} (A \backsim B)}{\sin \frac{1}{2} (A + B)}$			
a, B, b	c	p	2	$\cos (c \backsim y)$	$= \dfrac{\cos a \cos y}{\cos b}$	1	$\tan y$	$= \cos A \tan b$

Data.	Sought.	Series.	No.	Function determined.	Analytical Formulæ for the part sought.	No.	Auxiliary.	Formulæ for the Auxiliary.
							AUXILIARY ARCS OR ELEMENTS.	
a, A, b	c	p	4	$\sin(c + y')$	$= \dfrac{\cos a \sin y'}{\cos b}$	3	$\cot y'$	$= \cos A \tan b$
			5		$c = x \underset{\backsim}{+} y$	8	$\sin B$	$= \dfrac{\sin A \sin b}{\sin a}$
			6		$\tan x = \cos A \tan b$			
			7		$\tan y = \tan a \cos B$			
			12		$\cos x = \dfrac{\cos b}{\cos Z}$	11	$\sin Z$	$= \sin A \sin b$
			13		$\cos y = \dfrac{\cos a}{\cos Z}$			
B, a, A	C	q	2	$\cos(C \backsim y)$	$= \dfrac{\cos A \sin y}{\cos B}$	1	$\cot y$	$= \tan B \cos a$
			4	$\cos(C + y')$	$= \dfrac{\cos A \cos y'}{\cos B}$	3	$\tan y'$	$= \tan B \cos a$

TABLE of the FORMULÆ of Oblique-Angled Syherical Trigonometry, Continued.

Data.	*Sought.*	*Series.*	*No.*	*Function determined.*	*Analytical Formulæ for the part sought.*	AUXILIARY ARCS OR ELEMENTS. *No.*	*Auxiliary.*	*Formulæ for the Auxiliary.*
B, a, A	C	q	5		$C = x \overset{+}{\underset{\backsim}{}} y$	8	$\sin b$	$= \dfrac{\sin a \sin B}{\sin A}$
			6	$\cot x$	$= \cos A \tan B$			
			7	$\cot y$	$= \tan A \cos b$			
			12	$\sin x$	$= \dfrac{\cos B}{\cos Z}$	11	$\sin Z$	$= \sin a \sin B$
			13	$\sin y$	$= \dfrac{\cos A}{\cos Z}$			
B, a, C	c	r	2	$\sin (c + x)$	$= \tan B \cot A \sin x$	1	$\tan x$	$= \cos B \tan a$
			4	$\cos (c \backsim x)$	$= \tan B \cot A \cos x'$	3	$\cot x'$	$= \cos B \tan a$
			5		$c = x \overset{+}{\underset{\backsim}{}} y$	8	$\sin b$	$= \dfrac{\sin B \sin a}{\sin A}$
			6	$\tan x$	$= \tan a \cos B$			
			7	$\tan y$	$= \tan b \cos A$			

Data.	Sought.	Series.	No.	Function determined.	Analytical Formulæ for the part sought.	AUXILIARY ARCS OR ELEMENTS. No.	Auxiliary.	Formulæ for the Auxiliary.
B, a, C	c	r	12		$\cos x = \dfrac{\cos a}{\cos Z}$	11	$\sin Z$	$= \sin a \sin B$
			13		$\sin y = \dfrac{\cot A \sin a \sin B}{\cos Z}$			
b, a, A	C	s	2	$\sin (C + x)$	$= \cot a \tan b \sin x$	1	$\tan x$	$= \cos b \tan A$
			4	$\cos (C \backsim x')$	$= \tan b \cot a \cos x'$	3	$\cot x'$	$= \cos b \tan A$
			5		$C = x \overset{+}{\backsim} y$	8	$\sin B$	$= \dfrac{\sin A \sin b}{\sin a}$
			6		$\cot x = \cos a \tan B$			
			7		$\cot y = \cos b \tan A$			
			12		$\cos x = \dfrac{\cot a \sin b \sin A}{\cos Z}$	11	$\sin Z$	$= \sin b \sin A$
			13		$\sin y = \dfrac{\cos A}{\cos Z}$			

PART IV.

PRINCIPLES AND EXAMPLES OF THE PRACTICAL CALCULATIONS OF TRIGONOMETRY.

CHAPTER I.

General Principles of the Calculations.

§ 100. It has already been said, that order, and appropriate arrangement, are qualities indispensable in all calculations; but trigonometric calculations have more especially need of them. There are besides particular methods, that are of special use in such calculations, although they are applicable in a greater or less degree to all.

§ 101. We have seen that the formulæ have been transformed into such as are adapted to the use of logarithms, from the elements to the final result. In order to obtain this object, recourse has frequently been had to what are called auxiliary angles. It must have been observed, that, by means of these, we are enabled to make use of the properties, or rather the different relations of the elementary trigonometric functions, as calculations already made, in which the relative proportion of the variation of these trigonometric functions, is all that remains to be calculated.

§ 102. There is moreover another artifice, that contributes in a high degree to uniformity in the arrangement of the calculation; and it is astonishing, that this has so frequently, and for so long a time, been neglected, although pointed out

by Napier, the inventor of logarithms, himself, in his Canon Mirificus. It consists in employing the arithmetical complements of the logarithms; by which the final calculation of a result, depending upon any number of logarithms whatsoever, is reduced to a simple addition.

The general principle of this method may be explained in the following manner.

If from any number whatever, say 783192, we wish to subtract an other, say 639178, it is evident that, if we subtract the latter from the round number of the unit of the next higher denomination of the decimal scale of notation, and add the remainder to the first number, we shall have the same result as if we had made the subtraction; with this difference: that we must reject the unit of the next higher denomination in the decimal scale, that has been thus introduced. In our example we have for the number resulting from the subtraction, which is called the arithmetical complement: 360822
which being added to the first number, or . . . 783192

we have for the sum 144014
after rejecting the unit of the denomination next higher, as above directed; and this is in fact the difference of the two numbers taken as an example.

The use of this method would in ordinary calculations demand a strict attention to the effect of the next higher decimal denomination introduced; as, for instance, if from 379126
we had to subtract 5492, using the complement,
we have to add 4508

which gives in result 373634
where a unit of the fifth denomination, which has been introduced in the complement, is to be rejected, it being the denomination next higher than the highest denomination in the ubstracting number.

§ 103. In logarithmic calculation this rejection becomes merely mechanical; for in them we have always the same number of figures, and the characteristic can never be uncer-

tain to the extent of ten; for this would occasion a difference in the result of ten places of figures in the natural number; a mistake that cannot arise in any given case; and if the result were to be a trigonometric function, it would become an impossible one.

In order then to apply this method in Trigonometry, we always assume a characteristic = 10, from which we deduct the logarithm that is to be subtracted; the complement thus obtained is then added to the logarithm whence the former was to be subtracted. This subtraction from a characteristic = 10 is easily made, as well from right to left, as from left to right, which order may be most convenient in writing; to do this, we suppose the last number to the right to be 10, and all the others 9, and take their complements accordingly; thus each number obtained is the complement to 9 of its corresponding number, except the last on the right, which is the complement to 10; for this, being the first and lowest denomination, borrows from the next higher one a unit, which makes it become = 10, and this same borrowing, extending throughout the series to the characteristic 10, makes all the others, and this characteristic itself, become 9.

To give an example in logarithms, let it be required to subtract from logarithm 5,3714298
the logarithm 3,2910463

The arithmetical complement of the latter is . 6.7039537
When this is added to the first, the result, after rejecting 10 from the characteristic, is . . . 2,0803835
which is exactly equal to the difference between the two given logarithms; and having a 10 to reject, and retaining the characteristic 2, gives three significative figures to the whole numbers, the rest being decimals.

Let us take for a second example, one in which the result does not afford a 10 to be rejected, and which is therefore a

proper fraction. Say that from the logarithm	2,7863214
we had to subtract the logarithm	6,2491308
The complement of which would be	3.7508692

Adding this complement to the first log. we obtain 6.5371906 which not furnishing a 10 in the characteristic to be rejected, indicates it to be the logarithm of a decimal fraction; and the characteristic being 6, indicates that the first significative figure of the corresponding decimal fraction is of the fourth place of decimals, or has before it 0,000.

§ 104. This method is besides already introduced in the logarithms of the trigonometric functions; we have there an augmentation of ten units in the characteristic, which corresponds to an assumed radius of 10,000,000,000, instead of unity; which last would make all the trigonometric functions decimals, and their logarithms consequently negative, a result which this system is intended to avoid. This higher characteristic is rejected in the results, as we shall hereafter see, and by that, the method of calculation has only one system.

In order to render the means of ascertaining the number of these supernumerary tens in the characteristic, easy by mere inspection, it is customary to place a simple point (.) after the characteristics that are augmented by 10; and a comma (,) after the characteristic of the logarithms of natural numbers that are not complements. It results from this :—That the number which corresponds to a logarithm whose characteristic is 9, or a less number, with a (.), is a decimal fraction. In order to determine its value, or, which is the same thing, the place of its first significative number, it is to be observed: that the characteristic 10, which corresponds to 0, would give the unit place, and therefore the number which 9 represents would begin with the first decimal place, or tenths; the decimal number whose characteristic is 8, would begin with the second decimal place, or hundredths, and so on, descending in the scale; so that the complement to 10 of

the characteristic will indicate the place of decimals held by the first effective figure; the preceding places and the unit place being always filled up with 0; for it is proper to begin every decimal number at the unit place of whole numbers, as well in the case of decimals as in that of whole numbers; for to begin with a (,) or a (.) renders it too easy to mistake this mark as an interpunctuation from the preceding phrase.

There are authors who make use of negative characteristics, which are the complements to 10 of the above arithmetic complements, leaving the logarithms themselves positive; but their use is not only embarrassing, as all additions of positive and negative quantities in the same sum are, but it leads to mistakes in the operation; they are therefore to be rejected.

§ 105. It has been seen, that the formulæ frequently require combinations of the elements by addition and subtraction, in order to obtain numbers or angles, whose logarithms or trigonometric functions (in logarithms) are to be employed in the calculation; a certain order in their arrangement is necessary in order to shorten the calculation itself, and render its verification easy.

In this arrangement all repetition is naturally avoided; if the logarithms serve for several results that are equally the objects of research, they are written in such a way as to be easily added to each of the other logarithms that affect them in the different results; and of the whole is made a single example of calculation, whose parts are added alternately to obtain the respective results.

§ 106. The logarithmic tables that are of most frequent use, have generally seven places of decimals; the degree of exactness obtained by this number of decimals is sufficient for almost every kind of practical calculation. For special purposes there are tables that have ten, and even fifteen, places of decimals; while in cases that require less exactitude, or when the number sought has but few figures, we may be satisfied with using no more than five places of figures. It

is to fit them in case of need for this double purpose, that the tables of Callet have a point after the fifth decimal, which saves the attention to, or counting of, the numbers of decimals that are used; but no attention is paid to this point when seven places of decimals are employed. (I may remark here also, that the same tables that give the logarithms of trigonometric functions to every ten seconds, with the differences simply, and are therefore adapted to decimal multiplication, are more convenient, in accurate calculations, where decimals of seconds are used, than the great tables of Taylor giving these logarithms for every second without any differences, which of course must be first obtained, before proportional parts can be taken.)

As for the manner of using logarithmic and trigonometric tables, taking proportional parts, &c. reference must be had to the instructions given upon this subject with every logarithmic table; it would be here a useless repetition, and a description of the several artifices that facilitate their use would be too long if given in detail; attention and reflection in practice will teach them to every able calculator.

§ 107. Let us now proceed to the examples of the calculations themselves. Instead of any explanation that would interrupt the course of calculation, the logarithms will be marked by certain letters, and the results by the algebraic expressions of the operation that these quantities have under-undergone, wherever there is a double operation, otherwise it is supposed, that the sum of the logarithm is taken, as far as not separated by a line. We shall besides point out the data, and place at the top of each calculation the analytical formulæ to be executed, with a reference to the series and number in the body of the work.

CHAPTER II.

Calculations of Plane Trigonometry.

§ 108. The following examples may suffice for the calculations of *Right Angled Plane Trigonometry*; as all the other cases give similar processes.

In the right angled plane triangle *ABC*, *figure* 1, given, *d*, and *h*, and *A* = ∟*R*; to find sine *B*.

Formula A, No. 1. $\dfrac{d}{h} = \sin B$

$d =$	758,3	log =	2,8798411
$h =$	1935,5	$A : C :$ log =	6.7132068
$B =$	23°. 03′. 54″, 7	log sin =	9.5930479

Given, *h*, and *B*; to find *d*, and *k*.

Formula A, No. 1 and 2.

$d = h \sin B$; $k = h \cos B$

$h =$	2235,0	log =	3,3492755	$= x$
$B =$	16°. 23′. 46″.	log sin =	9.4506865	$= y$
		log cos =	9.9819694	$= z$
$d =$	630,89	log =	2,7999620	$= x + y$
$k =$	2144,10	log =	3,3312449	$= x + z$

Given, *d*, and *k*; to find tangent *B*.

Formula A, No. 3. $\dfrac{d}{k} = \tan B$

$d =$	31462,	log =	4,4977863
$k =$	94723,	$A : C :$ log =	5.0235446
$B =$	18°. 22′. 25″, 6	=	9.5213309

Given, d, and B; to find h.

Formula A, No. 1.

$$\frac{d}{h} = \sin B; \quad \text{gives } h = \frac{d}{\sin B}.$$

d	=	630,89	log	=	2,7999620
B	=	16°. 23'. 46" A : C :	log sin	=	0.5493135
h	=	2235,0	log	=	3,3492755

§ 109. The calculations of oblique angled plane triangles will follow here in the order of the problems; and are applied to a triangle, *ABC*, *figure* 6, *or* 7, whose sides, *a*, *b*, *c*, are respectively opposite to the angles of the same name.

§ 110. *Problem* 1. Given, *B*, *C*, *a*; to find *b*, and *c*.

Formula Y, No. 1.

$$b = \frac{a \sin B}{\sin A}; \quad c = \frac{a \sin C}{\sin A}$$

a	=	3745,8	log	=	3,5735446	=	x
A	=	61° 54' 25" A : C :	log sin	=	0.0544409	=	y
B	=	59. 58. 40.	log : sin	=	9.9374334	=	z
C	=	58. 06. 55.	log : sin	=	9.9289653	=	w
b	=	3676,37	log	=	3,5654189	=	$x + y + z$
c	=	3605,38	log	=	3,5569508	=	$x + y + w$

§ 111. ***Problem* 2. Given, a, b, C; to find A, and B.**

Formula Y, No. 2.

$$\tan \tfrac{1}{2}(A \backsim B) = \cot \tfrac{1}{2} C \, \frac{a \backsim b}{a + b}$$

$$\begin{aligned}
a &= 4901,6 \\
b &= \underline{3620,25} \\
a + b &= 8521,85 & A:C:\log &= 6.0694662 \\
a \backsim b &= 1281,35 & \log &= 3,1076677 \\
\tfrac{1}{2} C &= 20^\circ\, 24'\, 10'' & \log\cot &= \underline{0.4295133} \\
\tfrac{1}{2}(A \backsim B) &= 20.\ 00.\ 39,2 & \tan &= 9.6066472 \\
90^\circ - \tfrac{1}{2} C &= \underline{69.\ 35.\ 50.} \\
A &= 91.\ 36.\ 29,2 \\
B &= 47.\ 35.\ 10,8
\end{aligned}$$

Given, logarithm a, logarithm b, and C; to find A, and B.

Formula Y, No. 3 and 4.

$$\frac{b}{a} = \tan Z \quad ; \quad \tan \tfrac{1}{2}(A \backsim B) = \cot \tfrac{1}{2} C \tan (45^\circ - Z)$$

$$\begin{aligned}
&& \log b &= 3,4720537 \\
&& A : C : \log a &= \underline{6.4833417} \\
z &= 42^\circ\, 03'\, 46'',2 & \log\tan z &= 9.9553954 \\
& \underline{45.} \\
(45^\circ - z) &= 2.\ 56.\ 13,8 & \log\tan &= 8.7101908 \\
\tfrac{1}{2} C &= 35.\ 17.\ 00 & \log\tan &= \underline{0.1502104} \\
\tfrac{1}{2}(A \backsim B) &= 4.\ 08.\ 50,2 & \tan &= 8.8604012 \\
90^\circ - \tfrac{1}{2} C &= \underline{54.\ 43.\ 00} \\
A &= 58.\ 51.\ 50,2 \\
B &= 50.\ 34.\ 09,8
\end{aligned}$$

§ 112. *Problem* 3. Given, a, C, b; to find c.

Formula Y, No. 7 and 8.

$$\tan Z = \frac{2 \sin \tfrac{1}{2} C \, (a\,b)^{\frac{1}{2}}}{a \backsim b} \quad ; \quad c = \frac{a \backsim b}{\cos Z}$$

$a = 4539,3$	$\log = 3,6569889$	
$b = 3745,37$	$\log = 3,5734947$	
	$7,2304836$	
$(a\,b)^{\frac{1}{2}} =$	$\log = 3,6152418$	
$a \backsim b = 793,93$	$A:C:\log = 7.1002178$	$\log = 2,8997822$
$\tfrac{1}{2} C = 24°. 45'. 50''$	$\log \sin = 9.6220438$	
2	$\log = 0,3010300$	
	$\log \tan Z = 0.6385334$	$A:C:\log \cos = 0.6497139$
$c = 3544,02$		$\log = 3,5494961$

Given as above.

Formula Y, No. 10 and 11.

$$\cos x = \frac{2 \cos \tfrac{1}{2} C \, (a\,b)^{\frac{1}{2}}}{a + b} \quad ; \quad c = (c + b) \sin x$$

$a = 1966,26$	$\log = 3,2936444$	
$b = 3746,25$	$\log = 3,5735930$	
	$6,8672374$	
$(a\,b)^{\frac{1}{2}}$	$\log = 3,4336187$	
$a + b = 5712,51$	$A:C:\log = 6.2431730$	$\log = 3,7568270$
$\tfrac{1}{2} C = 29°. 24'. 15''$	$\log \cos = 9.9401069$	
2	$\log = 0,3010300$	
	$\log \cos x = 9.9179286$	$\log \sin = 9.7489740$
$c = 3204,8$		$\log = 3,5058010$

113. ***Problem* 4.** **Given, a, b, c; to find B.**

Formula Y, No. 15.

$$\sin \tfrac{1}{2} B = \left(\frac{(p-a)(p-c)}{a\,c}\right)^{\frac{1}{2}} \quad ; \quad p = \frac{a+b+c}{2}$$

$b =$	1920,6		
$a =$	3409,3	ar.co.log $=$	6.4673348
$c =$	2591,6	ar.co.log $=$	6.5864320
	7921,5		
$p =$	3960,75		
$p - a =$	551,45	log $=$	2,7415061
$p - c =$	1369,15	log $=$	3,1364510
			18.9317239
$\frac{1}{2} B =$	16°. 59′. 49″,5	log sin $=$	9.4658619
$B =$	33°. 59′. 39″.		

Given as above.

Formula Y, No. 16.

$$\cos \tfrac{1}{2} B = \left(\frac{p\,(p-b)}{a\,c}\right)^{\frac{1}{2}}$$

$b =$	2587,4		
$a =$	2468,8	ar . co . log $=$	6.6075141
$c =$	1584,2	ar . co . log $=$	6.8001900
	6640,4		
$p =$	3320,2	log $=$	3,5211642
$p - b =$	732,8	log $=$	2,8649855
			19.7938538
$\frac{1}{2} B =$	37°. 56′. 00″.	log cos $=$	9.8969269
$B =$	75°. 52′. 00″.		

Z

Given as above.

Formula Y, No. 17.

$$\tan \tfrac{1}{2} B = \left(\frac{(p-a)(p-c)}{p(p-b)}\right)^{\frac{1}{2}}$$

$b = 2325,2$
$c = 3106,4$
$a = 2459,8$

	$7891,4$		
$p =$	$3945,7$	ar . co . log $=$	6.4038759
$p - b =$	$1620,5$	ar . co . log $=$	6.7903510
$p - c =$	$839,3$	log $=$	2,9239172
$p - a =$	$1485,9$	log $=$	3,1719896
			19.2901337
$\tfrac{1}{2} B =$	$23^\circ. 49'. 41'', 1$	tan $=$	9.6450668
$B =$	$47^\circ. 39'. 22'', 2$		

§ 114. ***Problem* 5.** Given, C, c, b; to find a.

Formulæ Y, No. 19, 20, 21.

$$\sin B = \frac{b \sin C}{c} \quad ; \quad \tan y = \left(\frac{c \cos B}{b \cos C}\right)^{\frac{1}{2}} \quad ; \quad a = \frac{b \cos C}{\cos^2 y}$$

c = 3106,4	ar . co . log =	6.5077426	log =	3,4922574				
b = 2468,2	log =	3,3923803	ar . co . log =	6.6076197			log =	3,3923803
C = 50°. 21′. 14″	log sin =	9.8864910	ar . co . log cos =	0.1951495			log cos =	9.8048505
	log sin B =	9.7866139	log cos =	9.8981809				
			=	20.1932075				
			tan y =	0.0966037	ar . co . log cos²			0,2041452
								0,2041452
a = 4032,01	. . . .						log =	3,6055212

§ 115. **Calculations of the surface of the triangle *ABC*, whose sides are *a, b, c*.**

Problem 1. Given, *B, C, a*.

Formula Z, No. 1 ; $S = \dfrac{a^2 \sin B \sin C}{2 \sin (B + C)}$

$B =$	52° 58′ 50″	log sin =	9.9022375
$C =$	64. 11. 10	log sin =	9.9543454
$B + C =$	117. 10. 00	ar.co.log : sin =	0.0507651
$a =$	2468,9	2 log =	{ 3,3925145 3,3925145
	2	ar.co.log =	9.6989700
$S =$	2462334,	log =	6,3913470

§ 116. *Problem* 2. Given, *a, C, b*.

Formula Z, No. 3 ; $S = \dfrac{a \cdot b \cdot \sin C}{2}$

$a =$	3007,2	log =	3,4781630
$b =$	2092,85	log =	3,3207381
$C =$	89°. 54′. 50″	log sin =	9.9999995
	2	ar.co.log =	9.6989700
$S =$	3146810,	log =	6,4978706

§ 117. *Problem* 3. Given, *a, b, c*; to find *S*.

Formula Z, No. 5.

$$S = (p\,(p - a)\,(p - b)\,(p - c))^{\frac{1}{2}}$$

$$p = \frac{a + b + c}{2}$$

$a =$	3330,4			
$b =$	2965,9			
$c =$	2325,3			
	8621,6			
$p =$	4310,8	log	$=$	3,6345579
$p - a =$	980,4	log	$=$	2,9914033
$p - b =$	1344,9	log	$=$	3,1286900
$p - c =$	1985,5	log	$=$	3,2978699
				13,0525211
$S =$	3359390	log	$=$	6,5262605

CHAPTER III.

Calculations of Spherical Trigonometry.

§ 118. AFTER what has been said of the methods of calculation in the preceding chapter, it is not considered necessary to enter into the detail of the actual calculation of the formulæ of Right Angled Spherical Trigonometry, that are contained in series b. It may be observed, that they all require no more than the addition of two logarithms of trigonometric functions, in a manner exactly analogous to section 108, with this difference alone, that all the factors are trigonometric functions. Hence it is also evident, that relations only are obtained, not absolute quantities, as is the fact; for as we have only functions resulting from the relations of lines, no absolute quantity, or lineal dimension, can be in the result. This is the great means by which the relations of the immense and immeasurable distances that astronomy calculates, are obtained. When it becomes necessary to indicate real determinate magnitudes, as, for instance, in relation to the earth, it is evident, that the radius, which otherwise forms no element of the calculation, comes into consideration. In that case, it is necessary to multiply any result or formula, prepared for this purpose, by the value of the radius, ex-

pressed in that kind of unity in which it is wished to obtain the expression; in the first power when a mere lineal dimension is desired; in the square when a surface is required; and in the cube when a solid. This is exactly analogous to what has been said (section 11) in respect to right angled plane triangles; and all the formulæ of series Y, and Z, are examples of the same principle, as observed in sections 58, and 65; it applies equally to all the formulæ that follow hereafter.

We may proceed to the calculation of the formulæ of Oblique Angled Spherical Trigonometry, which require, of course, more arrangement and attention. As they are all expressly formed so as to admit of calculation by logarithms throughout, we shall dispense with the notation *log* before the trigonometric functions named; and consider it as always understood, that the logarithm of the trigonometric function indicated is used.

§ 119. *Problem* 1. Given, b, B, c; to find C.

Formula g ; $\sin C = \dfrac{\sin B \sin c}{\sin b}$

b =	80° 41′ 45″	ar. co. sin	=	0.0057515
c =	79. 40. 09.	sin	=	9.9929018
B =	83. 39. 59.	sin	=	9.9973412
C =	82. 13. 49.	sin	=	9.9959945

§ 120. *Problem* 2. Given, B, c, C; to find b.

Formula h ; $\sin b = \dfrac{\sin B \sin c}{\sin C}$

C =	40° 51′ 16″	ar. co. sin	=	0.1843305
B =	29. 14. 12.	sin	=	9.6887918
c =	39. 10. 04.	sin	=	9.8004375
b =	28. 08. 14.	sin	=	9.6735598

§ 121. *Problem* 3. Given, a, b, c; to find A.

Formula i, No. 1

$$\sin \tfrac{1}{2} A = \left(\frac{\sin (p-c) \sin (p-b)}{\sin b \sin c}\right)^{\frac{1}{2}} \quad ; \quad p = \frac{a+b+c}{2}$$

$a =$	73° 39′ 59″		
$b =$	84. 09. 58.	ar . co . sin =	0.0022551
$c =$	60. 15. 13.	ar . co . sin =	0.0613652
	218. 05. 10.		
$p =$	109. 02. 35.		
$p - c =$	38. 47. 22.	sin =	9.7968935
$p - b =$	24. 52. 37.	sin =	9.6239464
			19.3844602
$\frac{1}{2} A =$	29. 29. 30,8	sin =	9.6922301
$A =$	58. 59. 01,6		

Given as above.

Formula i, No. 2.

$$\cos \tfrac{1}{2} A = \left(\frac{\sin p \sin (p-a)}{\sin b \sin c}\right)^{\frac{1}{2}}$$

$a =$	98° 42′ 03″		
$b =$	83. 32. 26.	ar . co . sin =	0.0027658
$c =$	45. 48. 03.	ar . co . sin =	0.1495004
	227. 02. 32.		
$p =$	113. 31. 16.	sin =	9.9623282
$p - a =$	14. 49. 13.	sin =	9.4078800
			19.5224744
$\frac{1}{2} A =$	54. 45. 16.	cos =	9·7612372
$A =$	109. 30. 32.		

Given as above.

Formula i, No. 3.

$$\tan \tfrac{1}{2} A = \left(\frac{\sin (p-c) \sin (p-b)}{\sin p \sin (p-a)}\right)^{\frac{1}{2}}$$

$$a = 89^\circ 14' 16''$$
$$b = 73.52.43.$$
$$c = 67.24.15.$$

	$230.31.14.$		
$p =$	$115.15.37.$	ar . co . sin $=$	0.0436497
$p - a =$	$26.01.21.$	ar . co . sin $=$	$0.3579096.$
$p - b =$	$41.22.54.$	sin $=$	9.8202487
$p - c =$	$47.51.22.$	sin $=$	9.8700887
			19.0918877
$\frac{1}{2} A =$	$19.21.03,15$	tan $=$	9.5459488
$A =$	$38.42.06,3$		

§ 122. *Problem.* 4. Formulæ k.

These formulæ having evidently the same form as those of the preceding problem, the arrangement for calculation is precisely similar; it is therefore unnecessary here to give any examples. The only difference between them is, that they use the cosines instead of the sines; and that the factors alternate between the formulæ for the sine and the cosine; and consequently appear in inverse order in the formula for the tangent.

§ 123. *Problem* 5. Given, b, c. A; to find a.

Formulæ 1, No. 2 and 3.

$$\tan Z = \frac{\sin \frac{1}{2} A (\sin b \sin c)^{\frac{1}{2}}}{\sin \frac{1}{2} (c \backsim b)} \; ; \quad \sin \tfrac{1}{2} a = \frac{\sin \frac{1}{2} (c \backsim b)}{\cos Z}.$$

$b = 89^\circ 14' 18''$	sin $=$	9.9999616		
$c = 17.07.15.$	sin $=$	9.4689198		
$b - c = 72.07.03.$		19.4688814		
$(\sin b \sin c)^{\frac{1}{2}} =$		9.7344407		
$\frac{1}{2}(b - c) = 36.03.31,5$	ar . co . sin $=$	0.2301690	sin $=$	9.7698309
$\frac{1}{2} A = 42.16.08.$	sin $=$	9.8277639		
	tan $Z =$	9.7923736	ar . co . cos $=$	0.0706257
$\frac{1}{2} a = 43.49.58,8$			sin $=$	9.8404566
$a = 87.39.57,6$				

Formulæ 1, No. 5, 6, 8, 9, 11, and 12.

These being all of the same form as the preceding, using only other trigonometric functions of the same data, they come under the same form of calculation, the other functions taking the place of those used in the above formula, each for each, in its respective place. It is on this account not necessary to give examples of them.

Given as above.

Formulæ 1, No. 8, 11, 14. No. 13 is calculated upon the same form.

$$\sin Z'' = \frac{\cos \frac{1}{2} A \, (\sin b \sin c)^{\frac{1}{2}}}{\sin \frac{1}{2}(b + c)}$$

$$\tan Z''' = \frac{\cos \frac{1}{2} A \, (\sin b \sin c)^{\frac{1}{2}}}{\cos \frac{1}{2}(b + c)}$$

$$\tan \tfrac{1}{2} a = \tan \tfrac{1}{2}(b + c) \cos Z'' \cos Z'''$$

$b = 89^\circ\ 14'\ 18''$	$\sin = 9.9999616$	
$c = 17.07.15.$	$\sin = 9.4689198$	
$b + c = 106.21.33.$	19.4688814	
	$(\sin b \sin c)^{\frac{1}{2}} = 9.7344407 = m$	
$\frac{1}{2} A = 42.16.08.$	$\cos = 9.8692220 = n$	
$\frac{1}{2}(b+c) = 53.10.46,5$	$\left\{ \begin{array}{l} \text{ar.co.sin} = 0.0966290 = p \\ \text{ar.co.cos} \quad 0.2223493 = q \end{array} \right\}$	$\tan = 0.1257203$
	$\sin Z'' = 9.7002917 = m + n + p;$	$\cos = 9.9370883$
	$\tan Z''' = 9.8260120 = m + n + q;$	$\cos = 9.9195002$
$\frac{1}{2} a = 43.50.00.$	. . . .	$\tan \frac{1}{2} a = 9.9823088$
$a = 87.40.00.$		

Given as above.

Formulæ 1, No. 15 and 16.

$$\tan y = \cos A \tan b$$

$$\cos a = \frac{\cos b \cos (c \backsim y)}{\cos y}$$

A a

$A = 84^\circ 32' 16''$	$\cos = 8.9785888$	
$b = 89.\ 14.\ 18.$	$\tan = 1.8763321$	$\cos = 8.1236295$
$y = 82.\ 02.\ 57{,}8$	$\tan y = 0.8549209$	$\text{ar.co.}\cos = 0.8591168$
$c = 17.\ 07.\ 15.$		
$c \backsim y = 64.\ 55.\ 42{,}8$	. . . .	$\cos = 9.6271077$
$a = 87.\ 39.\ 57{,}5$	. . . .	$\cos = 8.6098540$

Given as above (but with one angle obtuse.)

Formulæ 1, No. 15 and 16.

$A = 121^\circ 36' 19'',8$	$\cos = 9.7193874 -$	
$b = 50.10.30.$	$\tan = 0.0788818$	$\cos = 9.8064817$
$y = 147.51.10.$	$\tan = 9.7982692 -$	$A:C:\cos = 0.0722788 -$
$c = 40.00.10.$		
$c \backsim y = 107.51.00.$	. . . .	$\cos = 9.4864674 -$
$a = 76.35.36.$	. . . .	$\cos = 9.3652279 +$

The effect of the obtuse angle at A, will be observed here, its cosine becomes negative; this is indicated by placing the sign $-$ at the end; in consequence of which also, tan y becomes negative; the obtuse angle is therefore to be taken for y, in consequence of which its cosine also becomes negative; and the angle $c \backsim y$ becoming again negative, the last calculation presents two $-$ signs, which producing again $+$, give for a an acute angle. This mode of accounting for the effect of the signs entirely obviates all difficulties.

The formulæ No. 17 and 18 being of the same form as the above, these examples will also serve for them.

§ 124. *Problem* 6. The formulæ of this problem, or series m, are all of the same form as the foregoing; the examples for calculation are to be arranged in the same manner, each respectively as its corresponding one.

§ 125. *Problem* 7. Given, C, a, b; to find A, and B.

Formulæ n, No. 1 and 2.

$$\tan \tfrac{1}{2}(A + B) \;=\; \cot \tfrac{1}{2} C \, \frac{\cos \frac{1}{2}(a \backsim b)}{\cos \frac{1}{2}(a + b)}$$

$$\tan \tfrac{1}{2}(A \backsim B) \;=\; \cot \tfrac{1}{2} C \, \frac{\sin \frac{1}{2}(a \backsim b)}{\sin \frac{1}{2}(a + b)}$$

$a =$	62° 25′ 32″		
$b =$	43. 19. 11.		
$a + b =$	105. 44. 43.		
$a \backsim b =$	19. 06. 21.		
$\frac{1}{2}(a + b) =$	52. 52. 21,5	ar. co. cos = 0.2192519	ar. co: sin = 0.0983805
$\frac{1}{2}(a \backsim b) =$	9. 33. 10,5	cos = 9.9939354	sin = 9.2199993
$\frac{1}{2} C =$	42. 05. 10.	cot = 0.0442502	cot = 0.0442502
$\frac{1}{2}(A + B) =$	61. 04. 00,7	tan = 0.2574375	tan = 9.3626300
$\frac{1}{2}(A \backsim B) =$	12. 58. 44,5		
$A =$	74. 02. 45,2		
$B =$	48. 05. 16,2		

§ 126. *Problem* 8. The formulæ o, No. 1 and 2, being exactly of the same form as the foregoing, the same example may serve as a type for them.

§ 127. *Problem* 9.

Formulæ p, No. 1 and 2.

$$\tan y = \tan b \cos A \quad ; \quad \cos (c \backsim y) = \frac{\cos a \cos y}{\cos b}$$

$A =$	32° 10′ 15″	cos = 9.9276086	
$b =$	57. 12. 03.	tan = 0.1908206	ar. co: cos = 0.2662444
$y =$	52. 43. 01,3	tan = 0.1184292	cos = 9.7822948
		$a =$ 46° 17′ 12″	cos = 9.8395099
$c \backsim y =$	39. 23. 48,9	. . . .	cos = 9.8880491
$c =$	92. 06. 50,2	or = 13. 19. 12,4	

The same formulæ, with an obtuse angle at A.

$A = 121.36.19{,}8$	$\cos = 9.7193874$ —		
$b = 50.10.30.$	$\tan = 0.0788818$	ar . co . cos $= 0.1935183$	
$y = 147.51.10.$	$\tan = 9.7902692$ —	$\cos = 9.9277212$ —	
	$a = 76°\ 35'\ 36'$	$\cos = 9.3652279$	
$c \backsim y = 107.51.00.$	. . . . . .	$\cos = 9.4864674$ —	
$c = 40.00.10.$			

Here the final result becomes a negative cosine, which therefore belongs to an obtuse angle, and produces c, acute, by the subtraction from the greater negative.

Given as above.

Formulæ p, No. 5, 6, 7, 8.

$$\sin B = \frac{\sin A \sin b}{\sin a} \quad ; \quad \tan x = \cos A \tan b$$

$$c = x \substack{+ \\ \backsim} y \quad ; \quad \tan y = \cos B \tan a$$

$A = 32°\ 10'\ 15''$	$\sin = 9.7262756$		$\cos = 9.9276086$
$b = 57.12.03.$	$\sin = 9.9245762$		$\tan = 0.1908206$
$a = 46.17.12$	ar . co . sin $= 0.1409781$	$\tan = 0.0195121$	$\tan x = 0.1184292$
	$\sin B = 9.7918299$	$\cos = 9.8949994$	
$x = 52.43.01{,}3$		$\tan y = 9.9145115$	
$y = 39.23.40{,}8$			
$c = 92.06.50.1$	or $= 13.19.12{,}5$		

Given as above.

Formulæ p, No. 5, 11 12, 13.

$$\sin Z = \sin A \sin b \quad ; \quad \cos x = \frac{\cos b}{\cos Z}$$

$$c = x \substack{+ \\ \backsim} y \quad ; \quad \cos y = \frac{\cos a}{\cos Z}$$

$A = 32°\ 10'\ 15''$	sin $= 9.7262756$		
$b = 57.12.03.$	sin $= 9.9245762$		cos $= 9.7337556 = m$
	sin $Z = 9.6508518$	ar . co . cos $= 0.0485393 = n$	
	$a = 46°\ 17'\ 12''$	cos $= 9.8395098 = p$	
$x = 52.43.01,2$			
$y = 39.23.49.$		cos $x = 9.7822949 = m + n$	
$c = 92.06.50,2$	or $= 12.19.12,2$	cos $y = 9.8880491 = p + n$	

§ 128. *Problem* 10. Here we have to repeat what has been said in problems 6, and 8; the formulæ of this problem, or of series q, take in calculation exactly the same form as those of *problem* 9; the examples of which, therefore, also serve for this problem.

§ 129. *Problem* 11. Given, B, A, a; to find c.

Formulæ r, No. 1 and 2, or 3 and 4.

$$\tan x = \cos B \tan a \quad ; \quad \sin (c + x) = \tan B \cot A \sin x$$

$a = 56°\ 13'\ 53''$	tan $= 0.1748021$		
$B = 60.42.08.$	cos $= 9.6896184$	tan $= 0.2509420$	
$x = 36.11.54,3$	tan $= 9.8644205$	sin $= 9.7712412$	
	$A = 50°\ 41'\ 15''$	cot $= 9.9132069$	
$c + x = 59.31.28,5$	. . .	sin $= 9.9354301$	
$c = 23.19.34,2$	or $= 95.43.22,8$		

Given as above.

Formulæ r, No. 5, 6, 7, 8.

$$\sin b = \frac{\sin a \sin B}{\sin A} \quad ; \quad \tan x = \tan a \cos B$$

$$c = x \overset{+}{\sim} y \quad ; \quad \tan y = \tan b \cos A$$

$a = 56°\ 13'\ 53''$	sin $= 9.9197521$	. . . . . .	tan $= 0.1748021$
$B = 60.42.08.$	sin $= 9.9405605$	. . . . . .	cos $= 9.6896184$
$A = 50.41.15.$ ar . co . sin $= 0.1114263$	cos $= 9.8017807$		tan $x = 9.8644205$
	sin $b = 9.9717389$	tan $= 0.4284988$	
$x = 36.11.54,3$			
$y =$ [illegible]6		tan $y = 0.2302795$	
$c =$ [illegible]	or $= 23°\ 19'\ 34'',3$		

Given as above.

Formulæ r, No. 11, 12, 13.

$$\sin Z = \sin a \sin B \quad ; \quad \cos x = \frac{\cos a}{\cos Z} \quad ; \quad c = x \pm y$$

$$\sin y = \frac{\cot A \sin a \sin B}{\cos Z}$$

$B = 60^\circ 48' 08''$	$\sin = 9.9405605$	$\sin = 9.9405605$	
$a = 56.\ 13.\ 53.$	$\sin = 9.9197521$	$\sin = 9.9197521$	$\cos = 9.7449501$
	$\sin Z = 9.8603126$	ar . co . cos $= 0.1619110$	ar . co . cos $= 0.1619110$
$A = 50.\ 41.\ 15.$	. . . .	$\cot = 9.9132070$	$\cos x = 9.9068611$
$x = 36.\ 11.\ 54{,}8$		$\sin y = 9.9354306$	
$y = 59.\ 31.\ 28{,}9$			
$c = 95.\ 43.\ 22{,}9$ or $= 23^\circ 19' 34'',3$			

§ 130. *Problem* 12. The formulæ of this problem having exactly the same form as those of *problem* 11, it is considered unnecessary to give examples of the calculation of series s.

THE END.

CORRECTIONS.

Page.	Line.	
15	16	After "called," add, or.
16	12 to 19	Between the fractions place a full stop (.) as sign of multiplication, instead of the comma (,).
—	21	"H" read, No.
17	penult.	in beginning, "3" read, 4.
18	10	Above "1 and 2" write, A, No.
26	1	"produced upon" read, produced on.
33	2	from below, $\frac{\sin a}{\sin b}$, read, $\frac{\sin a}{\cos a}$
35	15	"No. 4 and 9" read, No. 1 and 9.
38	3	Above "No. 1" in the margin, place, I.
39	5	"7" read, 8.

Page 54, at the bottom, add the following.

The formulæ 6, 7, and 8, give also, when divided by sine, or cosine, the following expressions for the tangent of the half angle by the tangent, and cotangent of the whole angle.

From No. 6:

$$\tan \tfrac{1}{2} a = \frac{(1 + \tan^2 a)^{\frac{1}{2}} - 1}{\tan a} = (1 + \cot^2 a)^{\frac{1}{2}} - \cot a \qquad 9$$

From No. 7:

$$\tan \tfrac{1}{2} a = \frac{\tan a}{(1 + \tan^2 a)^{\frac{1}{2}} + 1} = \frac{1}{(1 + \cot^2 a)^{\frac{1}{2}} + \cot a} \qquad 10$$

From No. 8:

$$\tan \tfrac{1}{2} a = \frac{(1 + \cot^2 a)^{\frac{1}{2}} + 1 - \cot a}{(1 + \cot^2 a)^{\frac{1}{2}} + 1 + \cot a} = \frac{(1 + \tan^2 a)^{\frac{1}{2}} + \tan a - 1}{(1 + \tan^2 a)^{\frac{1}{2}} + \tan a + 1} \qquad 11$$

Page.	*Line.*	
58	9	"$4 \cos b \quad 3$" read, $4 \cos b - 3$.
70	9	"$\sin^3 a$" read, $\sin^2 a$.
72	16	"$n \ldots\ldots n_8$" read, $n \ldots\ldots n_{10}$
75	13	**The divisor of the fourth term read thus, 2.3.4.5.6.7**
86	5	In the divisor, "$(a \backsim c)$" read, $(a \backsim c)^2$
93	3	"BC" read, B, C.
103	11	"Pcd" read, PCD.
107	26	"Lemma 1" read, Lemma 1 and 2.
——	27	"bc" read, be.
108	16	"$2 r^2 \pi \llcorner R$" read, $2 r^2 \pi 2 \llcorner R$.
111	12	"DGF by EGF" read, EGF by DGF.
——	last	"DGF" read, EGF.
118	8	In the divisor, "$\cos c_{\prime\prime} \backsim \cos c_{\prime\prime}$" read, $\cos c_{\prime\prime} \backsim \cos c_{\prime}$
119	1	" " "$\tan \frac{1}{2} C_{\prime\prime} \backsim C_{\prime})$" read, $\tan \frac{1}{2} (C_{\prime\prime} \backsim C_{\prime})$
123		Above the numbers in the margin, place, f.
126	4	"No. 14 and 15" read, No. 15 and 16.
130	14	"h" read, i.
——	—	"i" read, k.
135	5	"8" read, 3.
141	13	In the divisor, "$\cos A$" read, $\cos^2 A$.
147	9	"4" read, 3.
149	19	Place the "1" two lines lower.
151	6	"21" read, 2.
156	7 & 8	"$(a\,c)^{\frac{1}{2}}$" read, $(a\,c)^{\frac{1}{4}}$
157	4	Place $\sin B = \dfrac{b \sin C}{c}$ in the lower line.
——	5	"20" read, 21.
——	—	"19" read, 20.
——	—	Place the auxiliary in the lower line.
163	last	$\dfrac{\cos A \cos y'}{\cos B}$ read, $-\dfrac{\cos A \cos y'}{\cos B}$
164	4	"$\cos A \tan B$" read, $\cos a \tan B$
——	9	"$\cos (c \backsim x)$" read, $\cos (c \backsim x')$
169	26	"6.7039537" read, 6.7089537.
172	27	"logarithm" read, logarithms.
175	9	"20. 00. 39,3" read, 22. 00. 39,3.

(I C)

A

POPULAR EXPOSITION

OF THE

SYSTEM OF THE UNIVERSE,

WITH PLATES AND TABLES.

BY F. R. HASSLER, F. A. P. S.

" Quanto potius Deorum opera celebrare,
" Quam Philippi aut Alexandri latrocinia."
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INTRODUCTION.

It might appear proper in a work of the nature of the present, whose object is to render a subject of science agreeable to the general reader, to begin by pointing out its general usefulness. But in the present state of civilization and knowledge, extended and cultivated wherever the European race of man has spread its families, we may be permitted to assume that it is unnecessary to make an eulogium of astronomy.

We daily witness the arrival in our ports of numerous vessels that interchange the products of the soil, of the industry, and skill of every quarter of the globe, and which are guided in their course by the practical application of this science. It would be, therefore, almost an insult on the understanding of our readers, to undertake to prove formally, that astronomy is of practical utility to man, and exerts an important influence on his wants and his enjoyments. Even the least instructed is aware that without its aid he would be ignorant even of his

own age, as we should without it have no chronology, and no calendar.

It is in this science that the human mind has exhibited its greatest capacity; in it the imagination, aided by the most profound calculations and combinations, has made advances the most honourable to the genius of our race, and has shown the dignity of our being more clearly than in any other branch of human knowledge. The boldest hypotheses have been confirmed by the most subtile calculations, executed with mathematical exactness, on the basis of observations made by the most piercing eyes, aided by the utmost perfection of the arts.

The commencement of the present century has been especially embellished by the most brilliant success: planets have been discovered, in consequence of theoretic views of their probable existence, of the laws of their motion, and even of catastrophes that have served to determine their present state; comets, the terror of darker ages, have been followed by the mind's eye through the whole of their course, and their returns to the sphere of actual vision determined and predicted; others have been detected within the limits of the planetary bodies, and approaching in the elements of their motions so near to them, as to make it doubtful where we are to place the line that distinguishes the two classes.

Among the fixed stars, systems have been discovered moving under laws, apparently, nay we may say positively, the same as those which govern our own system. Such results have united more closely to our system those bodies which appear strangers to it, and extended the laws of it to infinite distances; they have rewarded the efforts of human talent, by bestowing gratifications the most lively, and the most elevated.

Although we do not wish to dwell longer on the physical interest of astronomy, still it will not be irrelevant to speak here of the moral influence this science exercises on those who cultivate it, and the elevation of mind with which it is calculated to inspire man.

Who is there that has not been struck with admiration at the sublime spectacle of a clear night, adorned with countless stars, that appear to grow in number the more attentively they are contemplated? But how much more elevated are the feelings and delights of him who in this profound silence discovers by the aid of his intellect, the laws of an everlasting movement. If in this state of enjoyment he be tempted to pride himself on his intellectual prerogatives, he is recalled to a just estimation of himself, by a comparison of the immensity and order that reigns in the celestial bodies with his own feebleness, and that of the efforts he is capable of

producing by the strength of his own means. He is thus led to appreciate the immense superiority of his intellectual energies over the mechanical powers that he can exert, of moral and mental enjoyments over those which are physical and corporal.

It would be useless to attempt to impress moral truth upon a heart directed by an empty head; neither preacher nor orator can ever effect this. The immutable laws of the mechanism of the heavens are the fullest and loftiest image of the immutability of the laws of morality, as well as that of their universal sway, and their inevitable influence.

The study of science in general represses the passions, but what study can do this so effectually, and at the same time withdraw the attention from the petty passions accompanying the business of every day life so completely, as that science which withdraws him wholly from the earth, and which does not permit him to consider even the entire mass of beings of his own species as the sole object of the solicitude, or as the only end of the organization of the great and eternal combination that he observes, and of which he calculates the movements.

Can it be conceived that a man, convinced that, were he even transported to another planet, he would find the same laws with which he has occupied his mind in this, the place of his birth, equally true and useful, would not be a good citizen in any country

to which he might be transported; he cannot be a stranger in any place; his moral dispositions, like his occupations, are not united to his nation, they fit him for being equally useful in every part of the globe.

A man accustomed calmly to contemplate the revolutions of the celestial bodies, on which depend the fate of worlds, sets a small price on the variations that fortune may cause him to undergo, and on the little masses that she has at her disposal. How small appear to him the greater part of the objects of human desire! In labouring to obtain his share of them, he cannot forget his moral position in regard to them, nor sacrifice his character to obtain his share of them. He may have need of the gifts of fortune, but he can never permit the desire of them to preponderate, or make of them his idols and his glory; they have no value with him but so far as they are applicable to the advancement of happiness, and to the promotion of the general good, that is evidently the object of all, however misunderstood it may be in individual cases.

The intrigues of power are despised by him, and even when obliged to submit to them, his mind is unconquered; as Galileo, when he was compelled to undergo the sentence of public recantation, for having taught the revolution of the earth, rose from his knees in saying, "for all this it turns," *(e gira nemeno)*.

His exit from this world is calm and fearless.—Accustomed to contemplate the great and immutable laws of nature, he obeys them without murmuring as without dread.

O quam contemta res est homo, nisi supra humana surrexerit.

Seneca quest. Nat. L. I.

PART I.

GENERAL DESCRIPTION OF THE SOLAR SYSTEM.

CHAPTER I.

General View of the Planetary System, and Laws of the Motions of the Planets.

§ 1. THE first subject which presents itself to our consideration in ASTRONOMY, because the most near and most interesting, is that system of celestial bodies, of which the *Earth*, our abode, forms a part; and which is called the *Solar System.*

The respective position of the planets, the laws of their revolutions around the same central body, the SUN, their mutual influence, and the appearances presented to a spectator placed upon the earth, are objects of inquiry the most interesting; and even the more detailed knowledge that we may be able to collect, in relation to their physical nature, presents a peculiar gratification, both from the success obtained in this inquiry, and in consequence of their evident relation to the earth in their general laws and organization.

§ 2. The primary planets which revolve around the sun, as far as they are known to us at present, are eleven in num-

be; they are solid opaque bodies, of a figure approaching the sphere. In the order in which they succeed each other from the sun, they are the following: Mercury, ☿; Venus, ♀; Earth, ⊕; Mars, ♂; Vesta, ⚶; Juno, ⚵; Ceres, ⚳; Pallas, ⚴; Jupiter, ♃; Saturn, ♄; Uranus, ♅.—(The marks added to the names being usually employed to designate them.)

Uranus was discovered by Herschel, in 1781; Ceres by Piazzi, in 1801; Pallas by Olbers, in 1802; Juno by Harding, in 1804; Vesta by Olbers, in 1807. All the others have been known from the most remote antiquity. The greater size, and the brilliant light, of these latter planets, rendering them conspicuous to the naked eye, they very soon attracted the attention by their seeming irregularity of motion among the numberless other stars, apparently fixed; and from this property of an apparently erring course, they derive the name of *Planets.*

The five smaller planets, discovered in our days, within the space of about twenty-six years, inferior in brilliancy, ranging even low in the rank of apparent magnitude, by which the stars were generally distinguished, escaped the distinct pursuit of the naked eye.—We owe this discovery to the greater perfection, and consequent increased power, of our means of artificial vision. What is still more remarkable, the discovery of the two planets, Juno and Vesta, was directed by the theoretical supposition of Olbers, that the two previously discovered, Ceres and Pallas, were too small to fill the rank of a planet, in their position in the Solar System; assigning, at the same time, the nodes of these as the proper place to look out for their fellow planets; they were therefore discovered by the application of theoretic principles, affording no small confirmation of the theories of astronomy, and manifesting the advanced state of the science.

§ 3. The more considerable of these planets present to us, on a smaller scale, systems of celestial bodies similar to

the Solar System itself, being accompanied in their revolution around the sun, by one or more satellites, that perform revolutions around their primary planets, in the same manner as these do around the Sun. Thus our Earth is accompanied by the Moon; Jupiter by four satellites or moons, similar to, but larger than ours; Saturn has seven, and, besides, presents us with the peculiar phenomenon of being surrounded, at some distance, by a double ring, flattened, and of comparatively small thickness: finally, Uranus has seven satellites.

§ 4. The discoveries of late years have also familiarized us with the comets, as parts of our Solar System, instead of extraordinary messengers of misfortune, as they were considered by our ancestors, who were unable to calculate their course and predict their return. Several of them hardly pass, in their farthest distance from the Sun, beyond the orbit of Jupiter, and perform their revolutions in less time than the larger, or remoter planets; it is, therefore, their smallness and want of light, not their distance, which renders them invisible to us during a part of their revolution. Of some, several returns have already been observed, and the influence of the proximity of the Planets upon their course, has been calculated. From the first, and then hardly credited, prediction of Halley, of the return of the comet of 1682, which was verified in 1759, further success was suspended, until Olbers discovered for the comet of 1815, an elliptical orbit, and a time of revolution only a few years less than that of 1759. The last decennium has shown us comets, the orbits of which approach so near those of the Planets at a mean distance from the sun, that we can refuse them the planetary rank only on account of their apparent physical constitution. Not a year passes without presenting astronomers with at least one or two comets; their varied courses in all parts of the immense space occupied by our Solar System, show a fullness and abundance, of which old systems of astronomy could not furnish an idea.

§ 5. All these planets and comets perform their revolutions around the sun in curves called *ellipses*, in such a manner that the sun always occupies one of the foci of them all. The great difference between the two diameters of this curve in cometary orbits has reduced their approximate calculation to that of the *parabola*. This curve deviates from the ellipse only by supposing the distance of the foci to be infinite, and is therefore appropriate, because we are unable to determine their distance; and because we see comets only in one of the parts of their orbit nearest to the sun.

These orbits are differently inclined towards each other; the planetary orbits, however, deviate much less from each other than those of the comets. In PLATE I. they are represented as projected upon the plane of the orbit of the earth, which is called the *Ecliptic*. PLATE II. represents the section of all these orbits in a plane perpendicular to the ecliptic, and to the equinoctial line, which forms in this a diameter: it represents, therefore, these inclinations as seen from a point at an infinite distance in the prolongation of this line through the point 0° ♈, from which it is generally agreed to count in stating the position of celestial bodies. TABLE I. annexed to this work, contains all the numerical elements of the orbits of the planets according to the most accurate and newest determinations, and also in approximate numbers for the ease of common comparisons. TABLE II. gives data for the individual natures of the planets in magnitude and other numerical determinations depending thereon. TABLE IV. presents the elements of the orbits of some of the principal comets. A comparison of the data of these tables will easily suggest to the mind a variety of circumstances and facts, which the longest description could with difficulty present.

§ 6. Though the orbits of the planets and comets are in their general form ellipses, these are, however, only approximations to their real course. The general and simple law, which guides the mechanism of all the celestial motions, oc-

casions deviations, in consequence of the mutual influence of the planets upon each other; these form the principal object of astronomy, as they form the main, or rather only cause of the great complication of appearances, and of the calculations that are necessary to render an account of them; by their various combinations it happens, that, strictly speaking, no celestial body ever returns to the same absolute point which it has once occupied; still this general law and its consequences enable us to calculate these positions to an astonishing degree of accuracy for a great length of time before, or after, any given moment.

§ 7. *Circular* figures were first attempted to be ascribed to the planetary orbits, as it appeared natural that the simplest curve we know should be that which nature had chosen; but here, as in many other parts of her great works, it has been proved, that what we should consider simple, has not always been the means which she preferred. Failing in representing the phenomena by direct circular motions, a combination of small circles revolving upon the circumference of greater ones, was attempted without success. It is useless at present to give an account of these exploded systems, which have vanished before that of *Copernicus*, confirmed by the law of *Universal Attraction or Gravitation*, inherent in all matter, which governs its mechanism in the most minute details, and has rendered us the most accurate account of all the phenomena which we observe. These old systems belong now only to the history of the science, whose scope it is to follow the uncertain steps of error as well as of truth, to show how to avoid the former and to attain the latter: here we intend to avoid the spectacle of human errors—to enjoy the contemplation of scenes and reflections elevating the soul of the reflecting man to much superior enjoyments.

§ 8. The law of *Universal Gravitation*, as simple in its general expression, as fertile and complicated in its consequences, is thus expressed:

All bodies whatsoever have a mutual tendency to approach each other with forces directly as their masses, and inversely as the square of their distances. In this shape it was expressed by *Kepler*, (de motibus Stellæ Martis, 1609,) long before the pure mathematics were so far advanced as to submit it to the test of accurate calculation, which *Newton* had the genius and good fortune to effect about eighty years later.

The revolution or motion of the celestial bodies, considered in itself, is a simple fact, which we have to record, and which we observe connected with this general law: the *primitive cause or impulse*, occasioning these motions, is unknown, as much as the cause of the gravitation itself; and both it would be useless for us to know. The simple, indivisible facts which nature presents to us, philosophy merely expresses distinctly; and thus accurately defined, they furnish the data for mathematical calculations.

§ 9. The special laws of these revolutions of the planets and comets, for the discovery of which we are also indebted to *Kepler*, are the three following:

A. *The planets describe ellipses around the sun, which occupies one of the foci of the curve.*

B. *The planets, in their course around the sun, describe sectors, whose arcs are proportional to the time employed to describe them.*

C. *The square of the times of revolution of the different planets, are as the cubes of their mean distances from the sun.*

To these laws the comets are equally subjected. *Kepler* deduced them at first from a combination of the observations: they have since then been proved to be general consequences of the law of universal gravitation.

We are also indebted to Kepler for a simple approximate expression for the relative distances of the planets from the sun, which may be thus stated. The distance of Mercury from the sun being expressed by the number 4, the following series will present the approximate distances of all the planets we are acquainted with:

PLANET'S NAME.	APPROXIMATE EXPRESSION.		
	By the law.	In single numbers.	
Mercury, ...	4	4	
Venus, ...	$4 + 3$	7	
Earth, ...	$4 + 3 + 2$	9	
Mars, ...	$4 + 3 + 2^2$	16	
Vesta, Juno, Ceres, and Pallas, taken together in a mean,	$4 + 3 + 2^3$	28	
Jupiter, ...	$4 + 3 + 2^4$	52	Pluto
Saturn, ...	$4 + 3 + 2^5$	100	
Uranus, ...	$4 + 3 + 2^6$	196	
The next planet that might be looked for in the system, if there should be any,	$4 + 3 + 2^8$	388	

This farther planet would therefore be at nearly double the distance of Uranus. These numbers, which at the time of Kepler it appeared almost presumptuous to mention, have received a singular confirmation by the modern discovery of five more planets, the four smaller ones of which verified the idea of Kepler that one more planet at least was to be found between Mars and Jupiter, if not lost by some catastrophe. And it was upon the hypothesis of such a catastrophe, that the discovery of the two last was made.

Uranus, though considerably larger than the earth, is from its distance hardly perceptible to us; little hope, therefore, can be held out, that we shall ever discover a planet belonging to our system at a distance more than double, which this law of their succession would appear to indicate as the nearest that can exist.

§ 10. The orbits in which the planets perform their revolution, (continuing the usual and convenient language of approximation,) are differently inclined to each other: they form planes intersecting each other in straight lines differently situated, but all passing through the sun as a point common to all these planes. (See Plate II.)

The lines of these intersections are called the lines of the nodes, and it is evident that every two planes will present one distinct from all the others. We, as inhabitants of the Earth, consider principally those formed by the intersection of each planetary orbit with ours, or with the ecliptic, as it has been called; the same mode of proceeding in their astronomy would be natural to the inhabitants of any other planet. It naturally follows, that each plane of the orbits, whether of the planets or comets, lies partly on one side of the ecliptic and partly on the other. Astronomy having been more cultivated in the northern hemisphere of the earth than in the southern, the language of appearance is even kept up so far as to call the part of a planetary orbit lying towards the north the *upper*, and that lying towards the south the *lower*, and therefore the point of the ecliptic where the planet will enter into the northern part of its orbit is called the *ascending node*, and marked thus, ☊, while the point directly opposite, or where the planet enters the part of its orbit south of the ecliptic, is called the *descending node*—marked ☋. These points, with the indication of the lines of intersection with the ecliptic, and also their reference to the general direction in space, as indicated by the subdivisions of the ecliptic, of which we shall soon speak, are seen in the first plate, at each place of the orbits and their references respectively, in the division of the margin.

§ 11. We can now perceive the necessity of a mode of indicating the positions, and, as we might call it, registering the places of the celestial bodies at any moment, in order to ascertain the particulars of their motion or rest. This has been done in the most ancient times by the division of the *ecliptic* into twelve signs, in the order in which the planets perform their revolutions, considering this plane indefinitely extended. In this order the signs are as follows: viz. Aries (♈), Taurus (♉), Gemini (♊), Cancer (♋), Leo (♌), Virgo (♍), Libra (♎), Scorpio (♏), Sagittarius (♐), Capricornus (♑), Aquarius (♒), Pisces (♓). It is usual, in

English, to preserve the Latin names, and to designate them by the signs here affixed to them. But a more convenient habit also prevails, to denote them merely by their number in the above order.

The general custom in mathematics of dividing the circumference into 360 degrees, gives to each of the signs 30 degrees. These signs corresponded anciently to certain collections of fixed stars, designated by the names of constellations, to which figures were given corresponding to these denominations. But the point of intersection of the protracted planes of the ecliptic and the equator of the earth from which these denominations begin with 0° ♈, being affected by a regular retrograde motion, (the subdivisions used by astronomers counting always from this point,) the coincidence of these divisions with their corresponding constellations is far from taking place now. This deviation amounts, at the present, to upwards of one whole sign, or thirty degrees.[1]

In determining the position of a planet, a comet, or a fixed star, the division just stated is used, without reference to the constellations, and has regard to this point of intersection such as it will be at the moment in question; therefore in designating a place, either future or past, this retrograde motion is accounted for either by the retrogradation that it will have acquired, or the advanced position which it still had at the time.

[1] Neither is the angle between these two planes constant, but varies within a limit of about one degree and one-third, in a revolution so slow as to require thousands of years to accomplish it. The determinations of our epoch in astronomy indicate a *diminution* of about fifty seconds of a degree in a century; this variation is therefore called the *diminution of the obliquity of the ecliptic.*

§ 12. To determine completely the position of a celestial body in space, we have still to assign to it the angular distance which it presents on either side of the plane of artificial ecliptic as above explained, in a circle perpendic

2

to this plane, and towards its pole, or the latitude north or south towards either of these poles of the ecliptic. Thus, by transferring to the celestial sphere our habitual language in assigning the situation of places on earth, we give to the celestial bodies longitudes counted from the point of the momentary intersection of the planes of our equator and ecliptic, and latitudes counted from the plane of the ecliptic towards either of its poles, north or south. In like manner, we can evidently determine the position of the orbit of any planet, by giving, beside the angle of inclination, the longitude of a certain point of the curve and that of one of the nodes: for the first it is usual to select the point in the greater axis, where the planet in its revolution is nearest to the sun—for the second the ascending node, as explained above. The lines 17 and 18 of TABLE I. give these determinations for the first day of the present century, which is that used in constructing PLATE I. and II.

CHAPTER II.

Positions and Revolutions of the Planets, and their general consequences.

§ 13. The results of the observations of centuries have enabled us to determine the elements of the orbits of the planets, and their positions in them according to the laws above stated. In the ten first lines of table I. are to be found the elements of the planetary orbits given with all the accuracy that has been at present obtained. To have the means of determining their distances, recourse is had to the unit which presents itself most naturally among them, namely, half the greater axis of the earth's elliptic orbit. In this unit and its decimal parts, all the distances which refer to the *orbits* of planets or comets are usually and most naturally given wherever reference is had to lineal dimensions. The comparison of the first and the second line of Table I. shows the difference between the two semi-axes of the ellipses described by the planets, which might be called their ellipticity: the fourth line, in giving the half distance of the foci in the same unit, serves equally to give an idea of the eccentricity of these orbits; while the third line, which expresses the same in a trigonometric function, is principally for astronomical use. The lines, 1, 2, and 4, it will easily be observed, are principally subservient to the mechanical description of the orbits. None of these orbits deviate very much from circles, for it will be seen that even of the largest orbit, that of *Uranus*, the two foci lie within the orbit of the earth, their distance being only about one-twentieth of the greater axis of the

orbit; of the other planets, *Juno* and *Mercury* have the greatest proportional ellipticity, and next to them are the other three newly discovered planets. Among the old planets, the greater eccentricity of Mars, joined to the facility of observing it, furnished Kepler with the principal means for his great discovery of the fundamental laws of all planetary motions, which still bear his name, (see § 9.) Mercury remains too long immersed in the light of the sun, and consequently presented difficulties to astronomers, until the introduction of greater optical powers enabled them to follow its course during the presence of the sun above the horizon. It will also be observed by the fifth line of TABLE I., as a distinction of the four newly discovered planets, that their orbits have greater inclinations than those of the ancient planets, and that the inclination of Pallas particularly exceeds by far all that was expected to be found for the orbit of a planet. The discovery of these planets, therefore, extended the limits formerly assigned to the zodiac, which had been considered as a belt of 9° in breadth on each side of the ecliptic, and which, divided into the signs of the ecliptic above described, formed the celestial houses of ancient astronomy; this may be seen in PLATE II.

§ 14. The sixth line of TABLE I. gives the time of a sidereal revolution of each planet, in days and decimals: this, though an individual measure taken from the peculiar phenomon of the rotation of the earth around its axis, is the only measure of time which nature presents to us—the seventh line contains the same quantity approximately in years, days, hours, &c. and all of which are either multiples or submultiples of the same unit. The sidereal revolution is the time which the planet employs to return to the same longitude in respect to the fixed stars; it is therefore independent of the retrogradation of the equinoxes mentioned above, which refers only to the earth's position in respect to its own orbit. The revolution of the earth is the measure of the year, and might therefore also be taken as a unit to measure the other revolutions by it; but as it does not contain an exact number

of days, it is rather more advantageous to use these in accurate indications of time. We shall hereafter see that even this unit admits varieties, and that various subdivisions of time result from the motions of the earth, the principles of which, as well as their application, will be explained in their proper place.

§ 15. The mutual influence of the planets upon their revolutions, as stated § 6, of course depend on the temporary combination of their positions, so that any certain effect is merely instantaneous. But by the continued effect of them all certain ultimate effects result upon the parts of the orbits, considered as their fundamental elements; these being only sensible after long periods of time, are generally called by the denomination of secular variations, though they are not in any way bound to such a period, which is only used as an easy means of accounting for their effect in ordinary astronomical calculations. These will be found in TABLE I., lines 12, 13, 14, 15, affected by the arithmetical signs of +, or —, to designate whether they act in augmentation or in diminution with the advance of time, counting their effect in the order of the signs of the zodiac. These small variations, therefore, show the slow and gradual changes which the whole system undergoes in process of time, and explain the necessity, in any representation of the state of the solar system, of adapting it to a certain epoch or temporary state. For instance, we have used for the plates and tables of this work the temporary epoch of the year 1800, except for the planets discovered since, for which the epoch of 1813 has been chosen. It may easily be imagined that the table cannot present any very precise results of the secular variations of these newly discovered planets; they have, however, already been attempted in the case of Ceres, which we have known for about a quarter of a century.

§ 16. The eighth line of the table shows by means of the greatest equation of the centre, the extreme deviations of the positions of the planets from those which they would occupy

in circular orbits described with an uniform motion, as was supposed in the old systems.

In the line of the greater axis of the orbit, that is, in the points of a planet's perihelion or aphelion, (which is also called the line of the apsides,) from which the places of the planets in their orbits are reckoned, this variation is $= 0$; because the two halves of the ellipses being equal, they correspond to equal times of the revolution, that is, each to half the revolution of the planet, considered as circular and uniform. From these two points, the extremes of the greater axis, the difference between the mean circular motion and the real planetary motion becomes greater as the angle made by the Radius Vector, (that is the line from the sun to the planet,) with the greater axis increases, until, the planet being in the end of the smaller axis of the ellipse, it attains the greatest magnitude quoted in the table for each of the planets. The great variation to which, by their proximity, the four small planets are subject, has not as yet permitted astronomers to determine this element of their calculations with a sufficient degree of accuracy, nor is it absolutely necessary.

§ 17. In comparing the whole revolution of a planet with the time it employs in performing it, we evidently obtain a mean angular velocity, which is given in the ninth line under the title of mean daily motion, and which would give the angle described by the planet in any given time by multiplication into the number of days and parts of days elapsed in the interval, if the motion of the planets were regular: it is, therefore, the mean situation given by this element that is reduced to the real by the application of the equation of the centre, and the equations that represent the influences of the attraction of the other planets.

To give an idea of the immensity of these velocities, compared with those that we are capable of producing on earth, the line 10 is added to the table, giving these velocities in feet for a second of time. Compared with the velocity of a cannon-ball, for which 2000 feet in a second is considered the

maximum, we see, for example, that while we consider ourselves at rest, we describe with the earth every second fifty times as much space in its orbit as a cannon-ball would, besides the effect of the angular motion caused by its rotation, which will hereafter become an object of our consideration.

§ 18. Before a system of the universe was formed capable of representing all the appearances under one law, and rendering them calculable, the synodic revolutions which are given in TABLE I., line 11, formed the principal, but irregular and uncertain, basis of the astronomy of the planetary motions. They represent the approximate common divisors of the times of the revolutions of the several planets, and the time of the revolution of the earth: they therefore bring about a return of the same series of appearances. But it is evident, that if these had been exactly regular, they would have given an accurate system by themselves. Their great oscillations, and the impossibility of reducing them to accurate epochs, renders them useless in the present state of astronomy, and no more than objects of curiosity.

§ 19. In this line has been placed, under the head of the earth, the duration of the mean solar year, which is the mean time of the return of the earth to the same position in relation to the sun, in like manner as the sidereal revolution presents the return of the earth to the same position in relation to the fixed stars. These two epochs, which are both called years, therefore differ from each other by the time which the earth employs to go through the part of its orbit comprehended between the point where it meets the same absolute point in the celestial sphere, to that where it meets the sun in the same position. As we have seen that the point from which we count the longitude of the celestial bodies, (namely, the equinox of the spring,) is affected by a certain motion, in an order inverse to the signs of the ecliptic; so we may, and really do, form a third kind of year of the return of the earth to this point, which, by the newest determination, is 365,242264 days, or 365*ds.* 5*hrs.* 48*min.* 51,6*sec.*, and is called the equinoctial or tropical year.

§ 20. The magnitude of the distances of the planets from the sun or from each other, convinces us of the futility of expressing them in the small lineal dimensions which we make use of on the earth, as leagues, miles, feet, &c. But another expression of them in lineal magnitudes of another kind appeared to me not to be uninteresting; namely, that in diameters of the sun and the earth. These, as introduced in lines 19 and 20, may serve as a kind of comparison between the distances of the celestial bodies and their magnitudes themselves, and in this way lead to interesting reflections. These numbers show, that it is impossible to represent in the same scale at once, both the sun, the planets, and their orbits, without exceeding the size admissible even in the largest atlas. For, giving to the sun only one-tenth of an inch as a diameter, the diameter of the orbit of Uranus would be about 35 feet, and the planets would still become microscopic magnitudes. Representing the earth under a diameter of one-tenth of an inch, that of the orbit of Uranus would become upwards of four thousand feet. In both cases the orbits of the satellites, and still more the magnitudes of the planets, would become indistinct. I have given in Plate IV. the proportional magnitudes of the sun and the planets, to give an idea of the preponderating magnitude of the sun as the central body of our system, which causes the great difference between the two above indications. These details belong to the individual description of the planets, which is naturally separated from the account of their revolutions round our central body, the sun.

§ 21. Considering the gradual increase of the distances of the planets from the sun, it is evident that for any different planet, different phenomena in all these movements must be the consequence, according to the situation or rank and distance which it occupies in this system, and these will form what might be called the astronomy of appearances for each of the planets, according to the position it occupies. The aspect of Plate I. shows, for instance, that to

Mercury, so near to the sun, and therefore also never receding far from the centre of the system, all the revolutions of the planets must appear as if performed around him, and small as he is, he may with more right than the inhabitants of the earth formerly did, consider himself as the central body —and it will be more difficult there to divest the inhabitants of this idea. If Uranus at the other extreme is, as it appears now to us, the last planet of our system, the more generalized aspect which this planet enjoys, places it rather in the situation of a spectator of the motions of our system than as a partaker of them. While viewed from Mercury, all the planets appear to pass rapidly from the direction towards the sun, losing themselves in its rays, to a direction diametrically opposite; Uranus sees all the planets only more or less rapidly approaching to or receding from the sun, on each side of it. Mercury, as seen from Uranus, hardly deviates one degree from the sun; and even Saturn, the farthest planet within Uranus, does not deviate above thirty degrees in the extreme from the sun, emerging from its rays only little more than Mercury does to us, and far less than Venus. (See TABLE V.) It must be difficult there to distinguish the motions of ten planets, revolving within the space of less than thirty degrees on each side of the sun, under inclinations of their orbits apparently deviating so little from each other in their plane, and the greatest number of them of so small diameter as to require optical means certainly superior to ours to become visible, considering that the sun itself must appear there under an angle of no more than about 1 1-4 minutes. (See TABLE II. line 3.)

For all the intermediate planets, these phenomena of appearances are mixed. Habit has called that planet nearer to the sun than a certain other one, its inferior, and that farther from the sun its superior one. The inferior planets, (as, for instance, Venus and Mercury in respect to the earth,) will appear to deviate from the sun on each side at an angle equal to that which the radius of their orbit will subtend at any time

from this planet. Thus, in table V. line 9, we find for Venus and Mercury, at their greatest distance from the sun, the angles 46° 19′ 49″ and 22° 46′ 28″ which, as the title of the table indicates, is the *mean* angle of its greatest deviation or elongation from the sun In like manner, all the other planets will see those that are nearer to the sun than themselves, at their greatest mean elongation or deviation, (the first being the usual astronomical term,) under the angles which are indicated for each of them in the table quoted; the first vertical column always presenting the planet observing, and the horizontal line opposite giving the greatest angle of the planet observed on either side of the sun, at the greatest distance from the sun.

§ 22. For the generality of the appearances of the planets, viewed from any one of them, it will be observed that the distinction of superior or inferior is permanent, but for the four smaller planets of newest discovery, the mean elongation of which will be found to approach nearly to a right angle, it may happen that this property alternates between them; so that they may become either inferior or superior, (or what is the same in relation to the sun, interior or exterior,) according to the place in their orbit in which they are at the time, and this phenomenon must be of peculiar interest on account of their small distance from each other at that moment. In general, we may easily see that the astronomy of these planets, in relation to each other, must present a considerable difference from ours.

§ 23. If we suppose a plane perpendicular to the ecliptic, passing through the sun and any one of the planets, and which therefore will always move with the planet, every other planet will pass such a plane twice in the course of its synodic revolution relative to this planet, namely, when the two planets are on the same side of the sun, and again when they are on the opposite side of the sun, relative to each other; in these positions the observed planet is said to be in *syzigy*. The inspection of plate I. easily shows that from any planet what-

ever any other planet may pass in a direction diametrically opposite on the other side of the sun, and this is called a superior passage. To an observer on earth, at least, these passages are always invisible, because the overpowering light of the sun deprives us of the sight of the planet.

In respect to the passages happening on the same side of the sun with the observing planet, it is evident by inspection of PLATE I. that a very essential difference must take place between the passage of a planet nearer to the sun, that is, an inferior one, and the passage of a planet remoter from the sun—that is a superior one. These latter passages are called *oppositions*, because the planet observed is in a position diametrically opposed to the sun, and these points serve in astronomy as fixed points of observation, because in them the earth, from which we observe, and the planet, correspond to the same longitude in the ecliptic, in their actual revolution, or as seen from the Sun. But the passage of an inferior planet between the observer and the sun, which constitutes a conjunction, furnishes the most instructive of these kind of phenomena, and therefore deserves our particular attention, as relates to the two planets Venus and Mercury, which pass between the sun and the earth. They furnish us periodically the interesting phenomenon of their apparent transit over the sun's disk, from which are derived some of the most important data of astronomy.

§ 24. These transits of Mercury and Venus, particularly the latter, furnish us a scale of the dimensions of the solar system: by their means the angle subtended by the radius of the earth, as seen from the sun, called the horizontal parallax of the sun, is determined: by this we are enabled to refer all the observations which we make on the surface of the earth to the dimensions or angles which they would present at the centre, and the proportion which we are thus enabled to form between the angles under which we see the diameter of the sun and that under which the diameter of the earth must appear in the sun, compared to their distance, enables us o de-

termine the magnitude of the sun and all the planets, in one and the same unit.

The figures 1 and 2, of PLATE V. may give an idea of such a transit. Fig. 1 shows the effect of the relative position of the sun and the two planets upon the visual line from the earth to the sun, by which the inferior planet appears projected upon the sun's disk, in the manner in which it is shown in figure 2. We have shown that the orbits of the planets are differently inclined to each other, and, from a comparison of the sidereal revolutions with the synodic, we conclude that their combination must occasion the inferior conjunctions to happen in very different points of the orbits of Venus or Mercury. It is evident that the conjunction must pass unnoticed when Venus in its orbit is in a greater apparent latitude than the semi-diameter of the sun, for then we shall lose sight of *Venus* as in a superior conjunction. The phenomenon of an observable transit of Venus over the sun's disk is therefore limited to such inferior conjunctions as happen in a sufficient proximity to one of the nodes of the orbit of Venus, so that the apparent deviation of Venus from the plane of the ecliptic, as seen from some one point upon the earth's surface, may be less than the apparent semi-diameter of the sun, as is represented in fig. 2. This occurs only twice in each century: the two last passages took place in 1761 and 1769. They have been assiduously improved by astronomers for the advancement of the science. The two next following in the present century will take place the 8th December, 1874, and the 6th December, 1882.

All the principles of reasoning and explanation used for Venus apply equally to *Mercury*, with the sole difference of their more frequent occurrence, as may be expected from the shorter synodic revolution of Mercury. They occur from 13 to 14 times in a century. The four last transits took place 1799, 1802, 1815, and 1822; the four next following will take place on the 5th May, 1832, the 7th November, 1835, the 8th May, 1845, and the 9th November, 1848.

From what has been said of the conjunctions and transits of Venus and Mercury, we may conclude the appearances which the earth itself must present to the planets remoter from the sun. In like manner, the appearances of the superior planets presented to us are exactly of the same nature as those which the earth presents to Venus and Mercury.

CHAPTER III.

Of the Satellites.

§ 25. The four largest of the planets are accompanied in their revolutions by one or more smaller bodies, performing revolutions around them in the same way that they themselves revolve around the sun. These bodies are called satellites, or secondary planets, and in contradistinction, the planets heretofore treated of are called primary planets.

They follow, in their revolutions around their primary, exactly the laws stated above for the revolution of these around the sun: the inclination of their orbits is likewise various, within small limits; their smallness compared with the other celestial bodies or primary planets, furnishes the greatest instance of the disturbing influence we have spoken of, and thus gives the astronomer means for the investigation of these disturbances.

§ 26. Of these satellites the earth has one, namely, the moon, which by its proximity appears to us so conspicuous, notwithstanding its comparative smallness, as to equal the apparent magnitude of the sun. But its defects in the great distinguishing properties of the sun, heat and light, did not long leave the attentive observer in doubt as to its secondary rank; and its phases or apparent changes of form, arising from its situation in respect to the sun, proved that all its light was borrowed from that body. As it performs nearly thirteen revolutions round the earth while the earth performs one around the sun, the simple feeling of propriety suggested the idea of its much greater proximity to the

earth. These revolutions furnish us with the type of the subdivision of time next in magnitude to the year, namely, the month or moon's revolutions. This is still used by many nations as a measure of time, though in a more perfect chronology, their duration has been made to vary from the actual revolution of the moon.

§ 27. *Jupiter* has four such satellites, all probably rather greater than our moon, though visible from the earth only by an eye armed with artificial optical powers: they were therefore entirely unknown before the discovery of the telescope. We find then here a complete system of celestial bodies, on a smaller scale, but similar to that of the primary planets.. Its distance from us prevents our observing the more minute consequences of the combination of their revolutions: those which we observe with advantage will be described in their proper place.

§ 28. *Saturn* has seven satellites, which, to be visible to us at their great distance, can hardly be less than the moon; but the nearer details of the phenomena they present have hitherto been imperceptible to us, and we even determine with difficulty the periods of their revolution. Their orbits have but a small inclination to each other. But this planet presents us with a phenomenon unique in our solar system; namely, at the distance of little more than its diameter it is surrounded by two belts, detached from the planet, of considerable breadth, and proportionally small thickness, which lie in a plane little inclined to those of the satellites—the only example of this form of matter in equilibrium in absolute space; all the other celestial bodies and planets, primary or secondary, presenting always a form approaching the spherical.

§ 29. *Uranus*, the farthest primary planet, has also seven satellites, that present the singular peculiarity of revolving around their primary in orbits nearly at right angles to the orbit of the primary: this must be productive of appearan very different from those that are seen from any of the

planets; but to observe and to discover the motions of these satellites is as yet the privilege of those astronomers only, who are furnished with the greatest optical powers; and our knowledge of them is far from being minute or precise.

CHAPTER IV.

Of the Comets.

§ 30. A LONG period of even advanced scientific improvement, elapsed before the comets could be proved to be other than accidental visitors, or even messengers of heaven with terrifying menaces from an irritated deity. All the great evils of humanity—war, persecutions, pestilence, &c., were attributed to them; to avert which, but a few centuries since, public prayers were ordered by governments on their appearance. From such fears the late discoveries of astronomy have forever and completely freed us. Hardly has such an unexpected visitor been visible for a few days before his path is determined: the theory of astronomy furnishes the laws which his motion follows, and a few observations give the numerical data whence to make an individual application to the case.

Though none of them is visible to us during its whole revolution, owing to their physical constitution, which we can hardly yet venture to determine upon, the orbits of several of them are entirely known, and with sufficient accuracy to determine the epochs of their return towards the Sun, when they become visible to us. So short even are the times of the revolution of some of them, and so fully inclosed are their orbits in the interior of the Solar System, that we could not refuse them planetary rank, notwithstanding their greater ellipticity, were it not for an evident difference in their physical nature; for, while the planets present solid spheric bodies of distinct magnitude, (and of course opaque,) the co

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mets present an undetermined shape; and fixed stars, which, by a planet's passage would have suffered occultation, have been seen through them. Some, indeed, have presented more or less of the appearance of a solid central part, denominated a nucleus; but besides the rather nebulous and gradually shaded-out light of the other part, the distinction between the two was never sufficiently precise to allow an accurate measurement authorizing the admission of a solid body.

§ 31. The number of comets that have been observed and their orbits determined, amount to more than 150; and a number of them have been calculated many times; but those whose return is known amount as yet to no more than six. In Table IV. only twelve have been collected, for some reason or other which the inspection of the table will easily show: they will, for instance, determine the limit within which comets have become visible to us. We observe that none has been seen, the perihelion of which was not within the orbit of Jupiter; the greatest number of them appear to pass, in the region of the Solar System between Jupiter and Mars, where the small planets have been discovered, as will be seen in Plate I. The greatest part of the orbits of those whose period is best known, lies in this region; but it is evident that, as we cannot see them unless nearer to us than Jupiter's orbit, an immense number of them may revolve beyond, as many of those that we see are so small, and often even difficult to see, that in earlier times of astronomy they may have passed unnoticed. The large comets are often accompanied by one, or even two, luminous streaks, which are called tails. While the frightened imagination of the gazers saw in this a sword, a bunch of rods, or some other instrument of chastisement, it may be expected that the accurate observation of these unfortunate meteors was not considered of interest, and is therefore but coarsely indicated; besides this, it requires peculiar means, which were not at the disposal of ancient astronomy: the data of

comets of those times are therefore scanty, as well in number as in accuracy.

In TABLE IV. has been collected the best data of the orbits of twelve of the most interesting, either because their returns are known, or for some other reasons stated, and the same have been introduced in PLATES I. and II. The full enumeration of all those whose orbits have been calculated, would be tedious and irrelevant: as the observations become more accurate, so we may hope to diminish their number by the discovery that some of the older orbits are repetitions of the new ones, which is naturally the manner in which these returns are best determined.

§ 32. The first instance of success in predicting the return of a comet is due to the astronomer Halley, who, in 1705, comparing together the orbits of different comets, discovered that the elements of the comets of 1456, 1531, 1607, and 1682, were so similar, that they might correspond to the same identical comet, and he predicted its return in 1758 or 1759, giving its revolution a period of about seventy-six years—(less than that of Uranus, which, however, was not then discovered)—and therefore made this period appear to exceed the then known limits of the Solar System.

The actual return of the comet to its perihelion the 12th March, 1759, confirmed the calculations of Halley, and constituted the first triumph of astronomy in this direction; in consequence, its next return may be expected in the year 1834 or 1835. The calculations of these large and very eccentric orbits, and the attractions under which the comets may come during their course by approaching the larger planets, the quantity of this effect being only hypothetically assignable, all concur to leave some uncertainty still in the accurate prediction of these returns; within these limits the comets recorded under the years 1006, 1080, 1155, and 1230, may even be referred to the same comet. In the better ascertained modern periods, we find that thirteen months more elapsed between the passages of 1531 and 1607 than

between those of 1607 and 1682, and eighteen months more between those of 1682 and 1759.

After a long interval of time and much exertion on the part of astronomers, the next success was obtained by Olbers in the comet of 1815, for which he determined an ellipse of about seventy-two and a half years; similar calculations discovered for the comet of 1812 a period little different, namely, seventy and a half years. But the calculations of the orbits of the comets being now habitually done, upon supposition of their being ellipses, and even the perturbations of their routes, from their accidental proximity to the larger planets, being taken into consideration, we were soon able to determine cometary orbits of very short duration; and the great accuracy of the observations upon which these elements could be grounded, has enabled us to follow several of them in their returns with a precision that, but a short time since, might have appeared almost unattainable, when we consider the short period of possible observations of this kind; the great velocity of the apparent angular motions of the comets (which exceed by far those of the planets;) and the incidental, variable, and as yet not actually determinable, attractions to which they are subject in different parts of their course, and which make them appear under somewhat altered elements, at every re-appearance.

§ 33. The elements, and the frequently observed return of the comet of November, 1805, render it peculiarly interesting; it proves itself to be the same as observed in 1786, 1795, and at different other times. Its revolution of only about 1213 days; the half greater axis of its orbit of only about 2¼ times that of the Earth's orbit, with an inclination of only about 13½ degrees to the ecliptic, assign to it a place completely planetary *between Mars* and the *four smaller planets of late discovery*, from which it is distinguishable only by its undetermined disk. For the calculation of this orbit, it is evident that the same means of astronomy are to be employed as for planets; for by its situation it is exposed to strong and varied

attractions. Since 1805, it has again been visible from the Earth in 1819—22—25, and will be so in January 1829, as TABLE IV. shows; together with the changes of the elements of the orbits in the different appearances.

As the orbit of the comet of 1770 could not be represented by a parabola, a mode that had been usual for the ease of calculation, (because ellipses of very great eccentricities, as the orbits of the comets were always supposed to be, approach very near to their corresponding parabolas in the neighbourhood of the perihelion,) the necessity of performing the calculation in an ellipse gave the first knowledge of comets of so short periods as in this case of about $5\frac{1}{2}$ years; an orbit almost entirely within that of Jupiter. And as it had at the same time been the nearest to the Earth of any comet, it became so much the more interesting. It appears that the attractions which this comet meets in its course, alter its elements so much as to render its discovery by them very difficult, or nearly impossible.

Similar to the foregoing in its principal phenomena, is the comet of 19th March, 1826. The period of its revolution, the places of its node and perihelion approaching to those of February, 1772, and December, 1805, they are considered the same, the orbit, though placed in a different direction from the foregoing, is very similar to it.

§ 34. The most ancient comet on the records of astronomy dates in the year 240 of our era. Its observations are due to the Chinese, and are very rude; the consequent elements cannot, therefore, be much relied upon. The comet of 1680 has approached the nearest to the sun of all; its orbit projected upon the ecliptic as in PLATE I., appears almost rectilineal to and from the Sun, on account of the great angle of inclination of its orbit with the ecliptic, of near 50°.

The largest comet yet observed was that of 1807. Still the diameter of its nucleus, as far as ascertainable, was not estimated at more than 1000 miles, while that of its nebuosity was 26037 miles; the volume of the nucleus therefore was only $\frac{1}{512}$ of that of the Earth.

The comet which has been observed in the greatest angular portion of its orbit is that of 1811, which was seen to pass through 185 degrees of its orbit in the proximity of the Earth.

Of all the comets hitherto observed, the most remote from the Sun in its perihelion is that of 1729. We might be authorised to conclude from it, that all comets that may pass outside of the orbit of Jupiter, will most likely forever remain unknown to us; and the great number of them which we find within these limits, may apprise us how much there might still be left to discover in the more distant regions, if our optical powers enabled us to obtain an insight into their phenomena.

PART II.

APPEARANCES OF THE PLANETARY MOTIONS, TO A SPECTATOR ON THE EARTH.

CHAPTER I.

General View of the Effect of the Eccentric Position of the Earth.

§ 35. The general exposition of the movements of the planets, their satellites, and the comets, in a birds-eye view, as it may be termed, such as we have hitherto presented, possesses great simplicity. One law, general and unique, producing a great and constant effect, and a combination of results, all consequences of the same law, and the individual system upon which it acts.

But the discovery of the order of this system, this admirable simplicity, producing these great effects, is the result of reflection, calculation, observation, in short, of the application of the faculties of man, the inhabitant of one of the planets, partaking of this motion, and occupying an eccentric position; from this, as the appearances are different, he was compelled to elevate himself to this exalted imaginary station by the efforts of his genius, a result as great as glorious for his intellectual faculties. Replacing ourselves in this eccentric

station, we soon find how man prejudices dictated by his senses he had to lay aside, in order to arrive at the great and general idea of a well organised system, which observation and calculation would present with equal accuracy. The more we enter into the detail, the more complicated we find the appearances; the more delusive; the more distant from the principles exposed; and the greater the difficulty of reducing them to the simple truth or reality. What the senses appear to dictate to man to consider as the most immoveable, the *Earth*, upon which he sees all apparently at rest, was to be submitted to a combined double motion imperceptible to our senses, yet of great velocity. And all this great apparent stability of the objects immediately surrounding us, was to be ascribed to their weight alone, and found to be a mere individual effect of the same general law which directs all the motions of the heavenly bodies in the immensity of space.

Our distance from the Sun is however so trifling in comparison to the whole system, that our position is even unfavourable from the comparative smallness of the diameter of the Earth's orbit, which must of course form the base upon which the measurement of these distances is grounded; hence it becomes necessary to proceed step by step in the inquiry, by repeating the operations, and making new combinations, thus securing accuracy by frequent approximations. This has actually been executed, by a zeal for intellectual improvement, to such an extent, that in passing by the inverse method from the results back to the appearances from which they have been deduced, we can proceed with ease and with certainty, and arrive at a satisfactory explanation of all these appearances. This method we have pursued in the preceding part, by a short exposition of the systematic results, from which we shall in this second part deduce the more detailed facts, that exhibit to us the explanation and the actual appearances.

§ 36. The task before us, then, is the individual astronomy of our Earth; that is, the exposition of the appearances of

this system from the planet the third in rank of distance from the sun; having, therefore, two planets performing their revolution within its own orbit, and all the other eight revolving outside of its orbit, such as has in general terms been quoted in § 21; and in doing this we shall gradually penetrate further into the details of the general system. We shall have only two principal phenomena to describe, but with such variations in the numerical results as the peculiar circumstances of each planet dictate. These two phenomena are; first: the appearances of the planets which revolve around the sun *within* the orbit of the spectator, or of the *inferior planets*, while the earth is itself also revolving around the sun; and, second: the appearances of the planets revolving at a *greater distance* than the earth, that is, of the *superior planets*, also combined with her motion in her own orbit. We shall therefore follow each of the planets during the course of one of its *synodic revolutions*, which, as we have shown, is the period of return of the same appearances, so far as they are assignable approximately.

CHAPTER II.

Apparent Revolutions of the Inferior Planets.

§ 37. Let us pursue the appearances *Mercury* presents to us during one of his apparent or *synodic revolutions.* When he begins to become visible in the evening twilight, he appears every subsequent evening farther from the setting sun, more elevated above the horizon, and remains longer visible, until he has reached a mean angular distance of about 22¾ degrees, as is indicated in Table V. for his greatest mean elongation, or apparent angular distance from the sun. During the time that Mercury appears thus receding from the sun, he has been farther from us than the sun, and was performing a part of his revolution in his orbit on the side of the sun opposite to the earth, his course appears to us as it really is, in the order of the signs of the Zodiac; it is therefore, as we have before explained, called *direct.* The different combinations of the distances of the two planets, the Earth and Mercury, from the Sun, in consequence of the eccentricity of their orbits, and the resulting inequality of the radii vectores, may reduce this angle to 16°, or bring it up to 29°. The velocity of the daily motions of Mercury appear during this time always diminishing as he deviated more and more from the direction of a perpendicular to the line of vision from the earth. In this point of greatest digression, the visual line from the Earth becoming a tangent to the orbit of *Mercury,* this planet enters into the part of its orbit whose convexity is turned towards the earth, to which it now approaches gradually by a motion which, as seen by us, is in the inverse order of the signs of the ecliptic, and is therefore called *retrograde.*

Shortly after this apparent change of direction, *Mercury* appears for a short time to remain stationary in respect to its revolution around the sun, because the *combination of the movements of Mercury and the Earth in their orbits*, produces a compensation, which causes the direction in which the motion of Mercury is observed from the Earth to remain parallel to itself, and therefore not to show any change in respect to the fixed stars, or the infinite distance to which they and the ecliptic are both referred. The apparent point of the orbit, and the time of the revolution in which this appearance takes place, are of course as variable as the greatest digression; the former varies between 15° 23′ and 18° 39′ of apparent angular distance from the sun. From this stationary state Mercury, continuing his course, and apparently approaching the sun, by the same apparent retrogade motion, becomes again invisible in consequence of its proximity to the sun, into which it appears to precipitate itself with an accelerated motion; for in this situation both approaching nearer the earth, and moving in a direction always more perpendicular to the visual line drawn from the earth to the sun, its apparent angular velocity increases, and becomes greatest while it is lost to us in the light of the sun, unless where it passes before his disk, as described § 23.

After having remained invisible for some time, Mercury re-appears to us in the morning twilight, receding each morning farther from the sun; his disappearance to the naked eye, on account of the increased brightness of the sun's light, taking place every day at a greater elevation above the horizon. The direction of his motion continues apparently retrograde, the velocity decreasing in consequence of the inclined direction of the motion to the line of vision from the earth, until it again appears stationary at an angular distance on the western side of the sun about equal to that at which it was so before its passage before the sun: having soon after attained its greatest elongation on that side of the sun, the apparent course becomes again direct, and approaches the sun with an

accelerated motion, in proportion as the visual line becomes again more perpendicular to the orbit, until his apparent proximity to the sun renders him again invisible for a short time; after which he becomes again visible in the evening, to renew the series of successive phenomena which have been described; within the limits, in time and apparent distance, that the combination of the momentaneous elements of Mercury and the Earth will occasion.

§ 38. The length of such an apparent revolution, *or synodic year of Mercury*, is, at a mean, 116 days. Mercury, therefore, presents this succession of phenomena about three times every year; this period, however, is subject to variation, as well as all the lesser phenomena it presents; this variation lies within the limits of 106 and 130 days.

The length of time during which Mercury is invisible on account of his proximity to the sun, must evidently, in addition to all the before-mentioned circumstances, be dependant on the power of vision applied to observe it, and the clearness of our atmosphere. It was therefore very difficult to determine the true orbit of Mercury with any degree of accuracy, before the eye was armed with telescopes, which now permit us to observe him by day at no very great distance from the sun.

The gradual approach and recession of Mercury during this revolution, is naturally accompanied by a proportional and successive increase and decrease of his apparent diameter; for every object of a determined and constant magnitude always appears under an apparent angle greater in proportion as the distance is smaller. However, in this part of the work, we shall continue to leave the magnitudes of the planets, and all the consequences resulting from them, out of view, in order to exhibit more simply and briefly their apparent course, referring all details to the third part, where it is intended to treat of each planet separately.

§ 39. Venus, the second of the two planets that perform their revolutions between the Earth and the Sun, presents in

apparent revolution, phenomena of the same character as those described in the case of Mercury. They are, however, of more interest, on account of her greater proximity of the Earth in the order of the Solar System.

A comparison of the times of the sidereal revolutions of the Earth and Venus in the 6th line of TABLE I. shows, that they are only about forty days different from each other. Their combination therefore presents a less frequent return of the same apparent positions in respect to the sun, which determine the synodic year. This is, at a mean rate, 584 days, and therefore brings an approximate return of the same phenomena only about six times in eight terrestrial years.

Following the *apparent course of Venus* in the same manner as we have followed that of Mercury, from her first becoming visible in the evening on the disappearance of the Sun below the horizon, we see her daily remaining longer after sunset, and appearing higher above the horizon, thereby pursuing her apparent course in the order of the signs of the ecliptic. As she recedes from the Sun, the velocity of her motion gradually lessens until she has attained the greatest eastern elongation, of (at a mean) 46° 19′ 49″. This, in consequence of the different combinations of the situation of the Earth and of Venus in their respective elliptic orbits, varies between 44° 57′ and 47° 48′.

From thence Venus again appears approaching the Sun by a retrograde motion, for having before been in the part of her orbit opposite to the Earth, and at the point of greatest digression, the visual line from the Earth having become a tangent to her path, she now continues her course on the same side of the Sun as the Earth, with increased velocity, in proportion both to her increasing proximity to the Earth, and the direction of her motion, which is now nearer to a perpendicular to the line of vision from the Earth to Venus, until her proximity to the Sun renders her again invisible to the naked eye.

During all this course, the brilliancy of Venus, which ren-

ders her of all stars the first visible at, or after, the setting of the Sun, has acquired for her the appellation of the *Evening Star*, from the most remote times. When, during this disappearance, the passage of Venus, between the Sun and the Earth, takes place in a part of the orbit sufficiently near to the nodes, the planet will be seen projected upon the disk of the Sun, and present what has been described above (§ 2 3) as a transit of Venus. This phenomenon, the peculiar proportion of the two Planets, Venus and the Earth, re-produces only twice each century; in all other cases Venus remains invisible during her inferior passage.

§ 40. Some time after this transit, or conjunction, Venus re-appears again in the morning, preceding the Sun in the same manner as Mercury; and shining with peculiar brilliancy; she is now called the *Morning Star*, and the continuance of her apparently retrograde motion occasions her to increase in distance from the Sun towards the west, to rise every morning earlier before the Sun, therefore to be longer visible. Her velocity constantly increases until she reaches her greatest elongation.

During her retrograde course Venus has been twice so situated as to show no apparent motion in longitude, or been stationary, for the same reasons as Mercury presented that phenomenon in its apparent course. Those stations occur when Venus is seen, at a mean, about 28° 51′ on either side of the Sun. The time between the two stations is about 96 days.

After her greatest western elongation, Venus, entering the part of her orbit most remote from the Earth, again resumes a direct motion, increasing in velocity in the same manner as it had decreased from her first appearance; appearing above the horizon every day later, and less high, she is lost sight of again on account of the proximity and the superior brilliancy of the Sun, during the time of her superior passage through the sizygy. After this she will again resume the same course of appearances or phenomena in her next synodic year.

§ 41. The similarity of the law under which these appearances of *Venus* and *Mercury* both recur, and the dependence of the phenomena upon the combination of similar chances, shews that they must be subject to similar variations, both in the angular magnitude under which each may appear, and the proportional time of appearance, and its duration; that therefore what was said in speaking of Mercury applies also to Venus, making due allowance for their different magnitude.

Notwithstanding this approximation to regularity of appearances, it is evident that no regular and constant motion could be formed of these appearances that could be referred to the Earth as a centre. For this reason, in all the systems that have been formed to explain the movements of the planets, there has always been this in common: that the Sun was taken for the centre of motion of these two inferior Planets, for no system has ever attempted to ascribe to any celestial body a mere oscillatory motion round any point whatever.

CHAPTER III.

Apparent Revolutions of the Superior Planets.

§ 42. In passing from the phenomena, presented by the revolutions of the planets which are nearer to the Sun than the Earth, to those of the Planets more remote from that body, we are prepared to observe a great change in all the appearances. It is immediately evident, that the Earth, performing her revolution around the Sun, within the orbits of these planets, they will appear to us successively in all directions of the circumference of the circle around the Earth, or in every point of the ecliptic, (as would be the astronomical expression.) Their apparent revolutions are subject to all the changes of direct and retrograde motions which have been observed in Venus and Mercury; and so much the more, as they are distributed over the whole circumference of the circle, as the position of the Earth, which constantly changes the point of vision, is in an orbit smaller than that of the farther planets and as the angle subtended by this orbit at the planet, the half of which is the greatest elongation, as shewn in Table V. is smaller. The discovery of the real revolutions of these exterior planets, from these appearances, was evidently to be made through that of the motion of the earth herself, and therefore required us to be freed from the prejudices arising from the apparent evidence of our senses: centuries of ignorance and subtle reasoning would rather give to all the celestial bodies, and the Sun itself, although admitted to be immensely greater than the earth, with the accompanying

planets, a daily rotation of inexplicable rapidity, than admit the simple and reasonable fact, that we, with the earth, share in the general laws of the movements and revolutions of all. To account for the frequent alternation of the motions from direct to retrograde, and inversely, the exterior planets were supposed to revolve in smaller circles, whose centres moved in larger ones forming the orbits. To make all centre upon the Earth, to which the appearances always refused to conform, the combinations were as varied as excuses invented to avoid admitting a simple truth always are. As none of these revolutions present a regular return, and lead the planets in never re-entering spirals, the astronomers of those times were so much perplexed, that Riccioli, to maintain the rest of the Earth, in contradiction to evidence, and his own conviction, resorted to the subterfuge of giving to each planet a peculiar angel to guide its arbitrary course.

§ 43. The first planet exterior to the Earth is *Mars*, of which we have already stated: that it furnished to the genius of Keptee the principal ground of the great discovery of the laws known by his name. To follow the appearances of *Mars* in the same manner as we have followed those of Mercury and Venus; he begins to become visible in the evening after sunset, appears gradually leaving the neighbourhood of the sun, in a direct course towards the east, making every evening a longer stay above the horizon; in this course he is first in the part of his orbit which lies on the opposite side of the Sun in relation to the situation of the Earth; his apparent motion taking place at first in a direction nearly perpendicular to the visual line from the Earth, and passing into a direction more inclined towards it, the apparent velocity is at first greater, and gradually diminishes, until Mars appears at an angular distance from the Sun of about 136°. At this angle he appears to have no motion in longitude, because this motion then exactly corresponds to that of the earth in its orbit, so as to render the visual lines parallel for a short time. They therefore show no alteration at the infinite dis-

tance to which the places of the celestial bodies are always referred.

After this, Mars, being in such a situation in respect to the Earth that their distance diminishes at the same time that the line of vision becomes more perpendicular to the relative motion, this last appears to increase rapidly, and being on the same side of the Sun with the Earth, he appears to move in the inverse order of the signs, or to be retregade. He passes in this course through the line, joining the Sun and the Earth directly opposite to the Sun, or 180° different in longitude; while in this point of relative situation the Earth presents to an observer in Mars the appearance of an inferior transit, as has been described in the case of Mercury and Venus. *Mars* is to us what is called in *opposition*, and in his greatest proximity to the earth; his apparent angular velocity is therefore also greatest.

From this point, as is evident from all that has been said before, the appearances return successively in an inverse order; the apparent motion continues to be retrogade until he again reaches an angular distance of about 136° from the Sun, when Mars appears again stationary, having employed in this retrograde course about 73 days, and described an apparent angle of retrogradation in the ecliptic of only about 16°. From this station Mars appears again to accelerate his approaches to the Sun in a direct motion, at a distance from the Earth always increasing, until, being in a position nearly opposite to the earth, on the other side of the Sun, he is lost again in the brightness of the Sun's light, and makes his passage of the superior syzigy unobservable with our present optical means. He becomes visible sooner or later after this, as the means of observing him are greater or less, and the series of the phenomena just described again recur.

The variability of the proportion between the radii of the orbits of the Earth and Mars, and the changes of the places in which those different phenomena occur in each apparent revolution, or synodic year, occasion all the epochs of time,

and the angular distance from the sun, in which they occur to be variable within certain limits, and as the apparent relative orbits are also incommensurable in parts of the ecliptic, they occur in different parts of it, as well as in different parts of the orbits.

§ 44. Next to Mars, as we proceed to a greater distance from the Sun, we meet the four smaller planets, *Vesta, Juno, Ceres* and *Pallas*. Their apparent motions, seen from the Earth, are of course subject to the same variations of direct, retrograde, and stationary, and are alternately accelerated and retarded, all in times and positions appropriate to the existing ratio of the elements of their orbits to those of the Earth, and the ratios of their sideral revolutions. The accurate details of those appearances which in the ancient astronomy were of interest, are entirely devoid of it in the present state of the science; therefore their detailed epochs were not even calculated. They must be very much similar to those of Mars. Here they may be passed over entirely, on account of the difficulty of ever seeing the planets themselves, in common life without better optical means than are usually met with.

The mean synodic year of Ceres is about 480 days, and from this the synodic years of the other three do not deviate much; they are all within the limit of 470 and 490 days.

These four planets are at distances from the Sun nearly equal, and at the same time the eccentricities of their orbits are so great, that they may alternately be each nearer to the Sun than the other; that is become inferior and superior alternately, which must produce phenomena of peculiar interest. In TABLE V. only the mean distances have been regarded, and the elongations inserted rather to fill up the table than as of any real utility. This complication of appearances renders these planets somewhat difficult to be distinguished from each other, and requires considerable discrimination. This was particularly the case in the beginning, when it was difficult to find them on their re-appearance after a superior transit.

§ 45. The changes of the appearances of *Jupiter* have a shorter period than the preceding, and this must always be the case the farther a planet is from the Sun; still however, their period can never become less than one of our years, just as we have seen that in the inferior planets the synodic year was more than one of *their* sideral revolutions or years. One revolution of the Earth in its orbit being contained nearly twelve times in a revolution of Jupiter, (see line 6 and 7 in TABLE I.) the synodic year of Jupiter is about 399 days. In consequence, the different appearances of this planet are repeated about eleven times in one of the sideral revolutions, or years, of Jupiter; but all the appearances cannot return in every year of the Earth, as they occupy somewhat more than thirteen months.

When this planet begins to be visible in the evening after sun-set, and recedes from the Sun, being longer visible every successive evening, he is evidently in a part of his orbit opposite to the then situation of the Earth relatively to the Sun, his motion is towards the east in the order of the signs, or direct. This direct motion continues until he has reached an apparent angular distance from the Sun of about 115½ degrees, when he begins to retrace back a part of his course by a retrograde motion, resulting from the circumstance that the angular motion of the Earth is greater than that of Jupiter, so that during about 121 days he describes an apparent retrogade arc, of only about 19° in longitude on the ecliptic; the apparent motion then becomes again direct: in this course Jupiter is always approaching nearer towards the Earth; this occasions an apparently accelerated velocity; and by an alternating succession of direct and retrograde motions, the effect of the combination of the distances and position of Jupiter and the Earth to each other, Jupiter arrives at a position directly opposite to the Sun, or 180° different in longitude. In this place he is to us in opposition, while the Earth presents to an observer in Jupiter the phenomenon of a transit or a mere conjunction with the Sun.

From this point Jupiter proceeds in his apparent course, approaching gradually towards the Sun on the opposite side from that on which he became first visible; the same phenomena succeed each other in an inverted order, by a gradual diminution of velocity, and alternating retrograde and direct motions, the same durations, angles, &c. as before the opposition, so far, namely, as the different situations of the Earth and Jupiter within their respective orbits admit these equalities. Still, however, he can never appear from the Earth to return to the same apparent place which he once occupied, and therefore never presents a regular re-entering course by his apparent motion, this being possible only by keeping an account of the effect of the motion of the Earth upon the appearances.

We omit speaking in this part of the effect of the apparent motions of the planets upon their satellites, it being more proper to treat of them in the 3d part, which is intended to give the details relating to each planet individually.

§ 46. The phenomena of the synodic revolution of *Saturn* are very similar to those of Jupiter. This period is about 378 days. The sideral revolution of Saturn, containing about $29\frac{1}{2}$ sideral revolutions of the Earth, the succession of appearances which constitute this apparent year, are repeated about 28 times in one of the years of Saturn, and one of them requires only about 12 days more than one of our years. The apparent motion of Saturn becomes retrograde when the planet is about $108\frac{1}{2}$ degrees to the west of the Sun, in his first digression from it, and the last period of his retrogradation, after the opposition, occurs at the same distance east of the Sun, when approaching towards it, presenting the same approximate alternation of direct and retrograde motion before and after the opposition in inverted order; the arc of retrogradation apparently described is in this planet $6\frac{2}{3}$ degrees, and the time employed in these retrogradation varies between 135 and 139 days.

§ 47. *Uranus*, the planet of our system most remote from the Sun, presents appearances very little different from those of Saturn, as they are consequences of the same principles, as they present similar geometric figures, and as the very great distance of both planets from the Sun acts to diminish this difference. This planet becomes retrograde at an elongation, or angular distance from the Sun, of about $103\frac{1}{4}^{\circ}$ east of the Sun, and has finished his retrograding appearances after the opposition, when it appears again at about the same angular distance on the west side of the Sun, he describes an arc of retrogradation of between $3\frac{1}{2}$ and 4 degrees, in which he employs a time of between 150 and 153 days. One of the sideral revolutions of Uranus being nearly 84 years and one month of the Earth, the synodic revolution resulting from this combination is about 369 days; not more than from three to five days longer than our year, and recurring more than 83 times in one of the years of Uranus.

This near coincidence between the synodic year of this planet and the sideral revolution of the earth, would have long kept astronomers in ignorance of the real course of Uranus, had not the fact of the revolution of the earth around the sun been admitted, even had it been possible that this body could have been discovered to be a planet, before the perfection of the theories and methods that are now at the disposal of modern science; by them it has been possible to trace back his steps to the observations made by several earlier astronomers, of stars then supposed fixed, and now entirely missing, with such certainty, as to make these observations subservient to the perfection of the elements of his orbit.

§ 48. Such, then, are the appearances of the revolutions of the planets around the sun, when observed from the earth, being itself engaged in its allotted course in this system; and leaving out of consideration the effect of the inclination of their orbits to that of the earth, or the ecliptic, as we are more used to call it. This inclination occasions their deviation from this plane in a perpendicular direction on either side. If this effect is taken into consideration, there is in

fact no point in which a planet appears totally immoveable; because the motion in latitude will still be apparent, even when no motion in longitude is perceptible, on account of its being compensated by a corresponding motion of the earth. Besides, as all these distinctions have their origin in old systems, derived from mere appearances, they are not susceptible of great accuracy, and the modern state of the science has rendered them useless, and therefore disregarded.

§ 49. The apparent versatility of the motions of the planets, joined to their superior and variable brilliancy, must naturally have struck every observer from the earliest epochs, and have excited the inquiry of the man of reflection. To his curiosity we are indebted for the science that most elevates the human mind, and one of the most useful in the extent of its results. It is this that guides the mariner over the trackless ocean, facilitates and secures that exchange of enjoyments, which so eminently distinguishes the present epoch of human history. Thus the whole history of mankind shows: that whatever tends to satisfy a desire of the human intellect, to improve the faculties and extend the sphere of its views, can never fail of ameliorating the state of human society; although it is not to be expected that the first observers and recorders of these phenomena could foresee in the speculations of their imagination the great results which have been their consequence.

This success is to be ascribed to the difference between astronomy and the other natural sciences; its whole foundation is purely mathematical; it is the strict mechanics of inanimate and inert matter; the bodies and phenomena being given, calculation and geometry can render an account of them in full. We can construct a mechanical imitation of nature in her great *Mechanics of the heavens*, while we cannot produce the smallest insect or plant of our own construction.

PART III.

INDIVIDUAL DESCRIPTION OF THE CELESTIAL BODIES COMPOSING OUR SOLAR SYSTEM.

CHAPTER I.

Of the Sun.

§ 50. THE two preceding parts have shown the proper revolutions of the celestial bodies of the solar system, and the appearances of their course as seen from the earth; that is, as the results of our observations present them to us; disregarding their figure, magnitude, and situation in respect to their orbits, we have been enabled to present these revolutions and appearances in a simpler and more concise manner. Our object must now be to describe them each individually, with its peculiarities, and what it has in common with the others; their proportional magnitude, their weight compared with each other, their form, and all the general properties which we are able to discover.

The first general observation that presents itself is: that these bodies all approach to the spherical form, which, under the supposition of an attractive power acting equally in all directions from a given point upon matter endued with equal propensity to obey this attraction, we may, upon mathematical principles, determine to be the form that they necessarily as-

sume. In the same principles of universal gravitation, we shall even find the cause and the determination of their small deviation from absolute sphericity, and the consequence that must necessarily follow it.

§ 51. The central body of our system, the sun, deserves our first attention, both for its magnitude, and because it announces itself immediately as the most important, from the preponderating influence which it exercises upon all the planets.

Of the elliptic orbits in which the planets and comets revolve, we have seen it to occupy a point common as a focus to them all, and from this point we shall see that it is in equilibrio with the whole system. Its apparent diameter, as seen by us, always surpasses half a degree, while that of the largest planet, Jupiter, in its greatest proximity to the earth, is less than $\frac{3}{4}$ of a minute.

The Sun is the apparent source of light and heat, is unchangeable in appearance, and always equal. The influence which it exercises upon the greatest, as well as upon the smallest objects in nature, is superior to all other influences which can come under the observation of man; directing the minute details of our life, as well as the greatest operations of nature on earth, constantly convincing us of our entire dependance on its presence or influence, and of its unrestrained rule over all we see; it is not astonishing, nay it is natural, that it should have furnished an object of adoration in the infancy of mankind, before science had rendered abstract ideas familiar. True religion can derive support from accurate knowledge alone; in its absence, and without the aid of science, idolatry, in one form or other, is unavoidable.

The more changeable motions and appearances of the moon, its total incapacity to influence sensibly the heat, or to affect any of the other more prominent phenomena of nature, have, notwithstanding the amenity of its variable light, and its apparent magnitude, never left the least doubt to an observing mind of its great inferiority in comparison to the sun. Ob-

servers were not long in discovering that this light was only a reflection from the sun, and not proper to the moon; the simplest comparison with surrounding objects gave the explanation: a tree illuminated on one side by the sun, appears so in a greater or less degree according to the place of the observer, and furnishes a parallel, as easy as satisfactory, to what is observed in respect of the moon; for in a clear night we see not only its illuminated but also its dark part. The moon will therefore furnish us with a separate subject, to be treated in its proper place, and we only mention it here to state its exclusion from all comparison with the sun as to importance in our Solar System.

§ 52. The revolution of the earth around the sun being performed in an ellipse, of which the sun occupies one of the foci, our distance from the sun varies within the limits of the whole distance of the two foci of this ellipse; the sun, therefore, also appears to us under a diameter, variable inversely with this change of distance. The greatest observed diameter of the sun is 32′ 35″ 5, or 1955″ 5, and the smallest 31′ 31″, or 1891″; the ratio of these apparent diameters is therefore also the inverse ratio of the two corresponding distances of the earth at its greatest proximity, or in perihelion, and at its greatest distance, or in aphelion;* for all intermediate distances the apparent diameter of the sun is inversely proportional to the distance of the earth from the sun.

§ 53. In comparing together the distances of the planets from the Sun, we took as our unit the distance of the Earth from the Sun. Now, to compare the magnitudes, diameters, and other dimensions or mechanical properties of the celestial bodies of our system, we take for the unit the diameter or other corresponding magnitude of the Earth; this evidently directly answers the purposes of astronomy. The expression of these in our small lineal or other ordinary magnitudes,

* The points of *Perihelion* and *Aphelion*, when reduced, as usual for solar tables and various other uses, to the appearances from the earth, are called *Perigee* and *Apogee*.

is a matter of mere curiosity, in which light only we may consider the data of the 24th and 25th line of TABLE II. The bases of this determination are the results of the transit of Venus over the disk of the Sun, which furnishes the most accurate means of determining what is called the parallax of the Sun, that is the angle subtended by the radius of the Earth as seen from the centre of the Sun; of this determination, however, it is not our object to give more than the result. The angle subtended by the diameter of the Earth is, when smallest, 16″ 7, when largest, 17″ 3; this is double the parallax of the Sun in its two extreme values. From it we obtain for the diameter of the Sun expressed in units of the Earth's diameter, 111,74; or the diameter of the sun is one hundred and eleven and about $\frac{3}{4}$ (near 112) times as large as the diameter of the Earth. The inverse ratio of the angles under which the two celestial bodies appear to each other is the ratio of their magnitude.

From this proportion between the diameters of the Earth and the Sun may be easily obtained the ratio of the surface of the Sun, equal to 12486 surfaces of the Earth, and its solid contents or volumes = 1395324,4 times the volume of the Earth.

From this fundamental comparison of the diameters, are derived all the diameters of the other planets, by means of their distances, and the angles under which we see these diameters at their different distances, an enumeration of which is to be found in TABLE II. lines 6 and 7, with the surfaces and volumes, thence deduced, continued to line 11. It may easily be concluded, from the smallness of the parallax which forms the foundation of these calculations, as well as of the angles of the planets seen from the Earth, that these determinations require great nicety, and a gradual approximation to the truth, and that the same high degree of accuracy cannot be expected in them as in the determinations of the places of the planets in their orbits.

§ 54. We observe in the different bodies that surround us,

that their mechanical effects, or powers of attraction, do not follow the exact ratio of their magnitudes. This mechanical effect, or gravitation, which we have seen to be the principle of all natural mechanical effects, and a general property of all matter, we conclude, from the comparison of an observation of a mechanical effect upon this principle : that the body exerting a greater mechanical effect, under an equal volume with another contains more matter; this quantity of matter, expressing *the mechanical effect of the body*, is what is called *its Mass ;* the *ratio of this mass to the volume* is called its *Density.* The mass is therefore the product of the density of a certain kind of matter into its volume.

As stated in the beginning, the celestial bodies act upon each other in their revolution in a ratio directly as their masses, and inversely as the squares of the distances ; we find in this principle, and the determination of the diameters, and consequently, of the volumes of the planets, the means of determining their masses, and consequently their densities, we thus in some measure absolutely weigh them, as we would other masses or bodies around us upon the earth. These results are given in the lines 12 to 15 of TABLE II. The mass of the sun is found to be 405871 times that of the earth ; but, in consequence of the much greater proportional volume, his density is no more than 0,2909 of that of the earth ; that is, the sun is not much more than about $\frac{1}{4}$ as dense as the earth, and its specific gravity is only a little more than water, or more nearly the same with that of resinous substances, namely, 1,3716 : the specific gravity of water being 1, while that of the whole earth is 4,715, as ascertained by the measurement of the mountain of Shehallion.

This low density of the principal body of our system is a peculiar phenomenon which shews : that its great preponderance must be due to the vastness of its comparative volume, and the consequent great quantity of the mass. Indeed, if we anticipate a little upon our future knowledge, by the mere assistance of the numbers contained in the TABLE

comparing in line 20, TABLE I. under the moon, with line 7, TABLE II. under the Sun; the distance of the moon from the earth being about thirty diameters of the earth, and the diameter of the Sun near 112 such diameters; we see that the whole revolution of the moon around the Earth, or our own little secondary system, might move in the interior of the Sun, with great facility, and abundant room. This is represented in a tangible manner in PLATE III. where, in one figure, the semi-circumference of the Sun is represented compared with the orbit of the moon, and those of the satellites of Jupiter, and, in the other, with those of Saturn with its satellites, also compared with the Sun.

The proportional apparent magnitudes, or what is usually called the disks of the Sun and the planets, which are dependant on the diameters, are also represented in PLATE IV. so as to shew the sum of all the diameters of the planets, compared with the diameter of the Sun, of which a part is represented upon the opposite part of the Radius. Summing up all the diameters of the planets, they form only about 0,26 of the diameter of the Sun, that is, little more than the fourth part. Extending a similar comparison to the other comparative magnitudes; the sum of the surfaces of all the planets is only $\frac{1}{55}$ of the surface of the Sun; the sum of all the Volumes, or bulks, only $\frac{1}{590}$ of that of the Sun; and, finally, the mass of the Sun, notwithstanding its density is only about one-fourth of that of the earth, is equal to 786 times the sum of the masses of all the planets and satellites, allowing each of these to have as much, or more mass than the moon.

§ 55. Small portions of matter, near the surface of the planets and celestial bodies, must evidently, in consequence of the general law of gravity, be attracted towards them with a power that will be very nearly constant all over the surface of the same body; at least no difference can be detected by common observations; this phenomenon is called the *fall of heavy bodies*, because to the individual observer

the approach to the center of the planet appears as a fall; this power will impress a certain velocity that is also constant, which it is usual to express by the space the heavy body will go through in the first second of the time of its fall. In line 21 of TABLE II. these heights of fall, upon the sun and the larger planets, are given as calculated from the fall observed upon the earth; for it is evident that these are calculable, because they must be proportional to the mass and the distance, that is the semi-diameter, of each planet; so, for instance, the table shows, that this fall being 16,089 feet upon the earth, it must be 424,75 feet upon the surface of the sun, and so for the other planets, as indicated by the table.

§ 56. Besides the phenomena of the *revolution of the celestial bodies around a center of gravity common to two and more*, as we have seen in the first part, we find them all affected by a *rotatory motion around an axis being one of the diameters of the body* itself, which is inclined at a certain angle to the orbit of their revolution. This motion, like the first, is a phenomenon given by nature, of which we have not to inquire the cause, but to describe and investigate the effects.

The time of one rotation of the sun around its axis is 25 days and about 16 minutes. It takes place in the same direction as the revolutions of the planets around the sun, and we are authorised to conclude from it, that the same cause and the same law direct both motions. Notwithstanding the complete solidity which we might ascribe to the sun and the planets, we observe their form to yield to the peculiar force that results from this rotation, which is called the *centrifugal force*, in consequence of which the planets assume an elipsoidal form, proportional to the *angular velocity* of this motion, in such a manner, that the parts which are most remote from the axis of rotation fly off from it more than those nearer to the same; thence *the parts about the two poles* become *flattened*, and the parts about the equator elevated; the proportional difference between the polar and the equatorial

diameter, therefrom resulting, is called the *ellipticity* of the celestial body.

In respect to the sun, we have to observe, in relation to this effect: that if its great diameter tends to increase its centrifugal force, its slow motion and small density both tend to diminish this effect. Indeed, the difference between the polar and equatorial diameter, which arises from this relation of the sun, is not observable to us; for according to theory it would amount only to 0,026, or $\frac{26}{1000}$ part of a second in the arc, a quantity of such minuteness that we cannot expect to determine it. The plane of the solar equator is inclined to the plane of the earth's orbit, or the ecliptic, at an angle of 7° $19\frac{1}{3}'$; the north pole of the axis being therefore inclined to the ecliptic 82° $40\frac{2}{3}'$, it lies in such a direction that its projection falls in the eighth degree of the sign Pisces, and the ascending node of the plane of the equator of the sun with the ecliptic lies in the longitude of about two signs and eight degrees.

§ 57. These data, as well as the time of rotation of the sun, are determined by the apparent motions of the spots which appear frequently upon the sun; but as they are neither constant, nor even always exactly of the same form during a whole appearance, the results obtained do not possess all the desirable accuracy; if these spots should be affected by a motion proper to themselves, they might even mislead us; but it appears from all the circumstances, that although accidental, they are phenomena occurring upon the surface of the sun itself. These spots are commonly visible from 12 to 13 days, and if their duration is sufficiently long for their reappearance, they become again visible after the lapse of from 14 to 15 days. The period of the rotation of the sun above quoted is an inference from the observation of a great number of such phenomena, taking into account the part of the revolution of the earth in its orbit performed during that time. The longest period through which it has as yet been possible to observe a spot in its different appearances has been about 70 days. Since the invention of the telescope

has furnished the means to make these kind of observations, which were before impossible, a great number and great variety in them has been observed, for this appearance is very frequent. In 1611, almost all the year spots were seen upon the sun. Scheiner, who devoted great attention to this subject, and has left a large folio volume of descriptions of them, saw as many as 50 spots at once. From 1650 to 1670, no spots were seen; while in 1785, 38 were seen at once, which were together equivalent in size to an eclipse of $\frac{1}{3}$ of the sun's diameter. Of the physical nature of the spots we are entirely ignorant, therefore conjectures in respect to them are the more numerous: it appears needless here to relate any of them.

§ 58. That property of the sun which is so essentially distinctive from all the planets, namely: that he acts as the *cause* or *source* of *Heat* and *Light*, has occasioned an idea that its constituent parts must be in their nature different from those of the earth or the planets, and its physical state of such a nature as not to admit of similar phenomena with them, or to permit the existence of animated creation upon its surface. Though it is evidently impossible, in principle, to decide by positive observations what may be the actual nature of the constituent parts of the sun, or the kind of animated beings to whose life it would be congenial; still we have seen that the same law of universal gravitation, with all its consequences, equally governs the great motions of the celestial bodies in the immensity of space, and the minute phenomena which surround us upon the earth; we may thence be allowed to enter upon the discussion of this subject by the aid of reasoning grounded upon the general principles of a sound philosophy, deduced from the phenomena which we observe around us.

Thus, 1st: With the idea of matter, the idea of its absence or presence is inseparable, or speaking numerically, its zero must be ascertainable. But all matter with which we are acquainted is endowed with, or contains, heat, and only differences of heat, not absolute quantities of it, are assignable;

exactly as we observe to be the case in gravity; upon exact and sound philosophical principles, then, we may, nay must, *refuse to heat or caloric the denomination of matter.*

2d, If we admit heat to be material, we must ascribe to it the property of being subject to gravity, as this property is common to all matter; this not being the case, recourse has been had to the term caloric, which is conceived to be an imponderable fluid, an idea composed of two things contradicting each other. Instead of this, we find heat in all the simple or compound bodies that come under our observation, in a quantity in some proportion to their mass or magnitude, though with variations in different kinds of matter; again entirely similar to what we see in gravity; like this, therefore, *we must admit Heat to be a Property of matter.*

3d. All bodies mutually communicate heat to each other within the sphere of their mutual influence, until an equilibrium is established between them, according to their respective capacities and powers; in the same manner that we see universal gravitation governing the whole planetary system in one equilibrated motion. And this property of communicating and absorbing heat is, under circumstances otherwise equal, and the same kinds of matter, either simple or compound, proportional to the quantity of it, in the same manner that we have seen gravity to act in proportion to the mass, and its capacity for gravitation, that is to say, the density.

4th. To prevent the action of gravitation, or the communication of heat, from affecting any particular mass or body, the effect must in both cases be equally absorbed, or what is the same thing, impeded, by a body receiving this effect instead of the protected one. When we wish to produce heat in its highest degree, that is fire, by artificial means, the operations are exactly analagous to the use we make of gravity in our mechanical constructions; as for instance, in the construction of clocks, by which the force and effect of this very gravitation is measured by an inverse process.

§ 59. These principles and considerations, extended in their application to the planets and the sun itself, present some general results, as to the physical constitution of the sun and the planets, which the observations confirm, as near as our present knowledge admits; as 1st: we obtain for the earth, as the sum of all matter composing it, a certain mean result of capacity and quantity of heat, as well as of matter itself, proportional to the quantity of this last, and the same must be the case in every planet, and the sun itself, with modifications only of the general law, as we observe on a small scale in the earth.

2d. The preponderating mass of the sun, compared with that of all the planets taken together, is of itself sufficient to account for its pre-eminent property of containing, causing, and communicating heat; otherwise we should be obliged to have recourse to the supposition of a peculiar kind of matter called *Heat* or *Fire*, and to conceive it to form the body of the sun, or even to emanate from it. Therefore an aggregate of different kinds of matter similar to that which we observe upon earth, or generally of the same nature as the planets, collected to that quantity, would of itself be sufficient to produce the whole effect observed.

3d. In all inanimate matter we find the density decreasing when the heat increases; the sun appears to give us the strongest example of all of the generality of this principle, in consequence of its presenting, with such a preponderance in volume and mass, a density of only about one-fourth of that of the earth. In the planets we likewise observe a diminution of density accompanying augmentation of mass and volume.

Though, therefore, the exact proportion in which the density diminishes with the augmentation of the mass is not known, the fact in itself is constant. It might then, perhaps, be permitted us to reverse the whole of this reasoning, and from these great facts in nature, to deduce the proof that *Heat is a Property of matter, contained in it in a certain determinate proportion to the mass.*

4th. If we assume the temperature observed in mines as the temperature of the earth, to which it at least affords an approximation nearer than that observed in the atmosphere, and that the matter be similar in its nature through the whole system, we find in this constant relation of the temperature to the mass of the sun, a degree sufficiently high for the intense heat which we there observe, and for all its consequent effects; and therefore for its light; for we know light to be in all bodies the result of a highly elevated temperature. The temperature of deep mines being about 56° of Fahrenheit's thermometer scale; that is about one-third of the scale of temperature between the freezing of water, at which vegetation is stopped, and the boiling point of water, at which animal and vegetable life are both destroyed. In the absence of an actual 0 of heat, we may adopt this scale, although imaginary; and by the direct simple ratio of the masses, we may conclude a temperature of 135,000 times the extent of the whole scale, from the freezing to the boiling of water, while our observatious give for the temperature at which *Ignition takes place on earth*, 650° of the same scale, that is only $3\frac{1}{3}$ times the same extent of the scale.

5th. We also find, by the observations of the phenomena of heat on earth, that bodies absorb or retain heat in an inverse proportion to their density; that is, the heat they contain becomes proportionally less sensible, as it is employed in maintaining the less dense state of the body, we therefore find in this a power capable of altering the effect of the heat that is sensible in the immediate vicinity of the body, whereby the sensible heat becomes different from that which would result from the general action of the same upon the two effects above stated, namely, the density and the specific heat; this appears analagous to the phenomena of gravity.

The same principles we observe as we rise in our atmosphere more distant from the solid part of the earth, which from its mass contains more heat, and from its density has a less

capacity, or absorbs less, the heat, the less do we find the influence of the sun or the earth, or the more the rarity of the atmosphere absorbs heat; for the diminution of sensible heat is great and rapid in the higher regions of the atmosphere so that at a certain height above the level of the sea, the effect of the sun of one summer is insufficient to melt the ice of the preceding winter, while the lands nearer to the level of the sea shore, or closer to the great mass of the earth, and in a denser atmosphere, are generally warmer.

§ 60. From these data and reasoning, we may conceive that we have the satisfactory explanation not only of the heat (and even of the light) of the sun, but, at the same time, of the law of the mutual influence of the heavenly bodies upon each other, in respect to heat, which is to be stated thus; namely: as in respect to Gravity so *in respect to Heat, different bodies, or portions of matter, act mutually upon each other in the direct ratio of their masses, and, inversely, as the square of the distances.* The last part of this statement is the unavoidable mathematical consequence of all effects proceeding from a certain point as a center of action, and is therefore a general law in this case, as well as in attraction, light, &c. &c.

Though this is not a proper place to enter into the more minute consequences of this law in other branches of natural philosophy, it may, however, not be improper merely to state that the phenomena are in no way in contradiction with it; thus, for instance in chemistry, the production of the great heat that often results from chemical mixtures, refers to cases where the denominator of the above ratio, that is the distance, may be considered as infinitely small, by which the value of the fraction is increased proportionally. The reflection of heat takes place upon these principles in the same manner as in heavy bodies. It is not the body acted upon, but the mechanical law that determines the reflection; in like manner, in heat, not the heat considered as matter, but its

effect, or law of communication with the matter acted upon by the heat, is the phenomenon offered in its reflection.

§ 61. With Heat at a certain degree of *Intensity*, which is here exactly analagous to *Density* in relation to matter, we always find light united, and following the same mechanical laws, though of course in different proportions in bodies of different nature; adhering to the general phenomenon, we have found the heat of the sun, when assumed to be in the direct ratio of the mass, to exceed that of Incandescence or Fire about 200 times. We thus find no cause for our being astonished at the great light of the sun; numerous phenomena on a small scale may be adduced in support of this; the greater brilliancy of the larger planets, the masses of which we may on the same ground consider as more comparable with the sun than the smaller ones, is equally in support of these principles; and, by this means, they approach to that degree of apparent light which is imparted to the smaller inferior planets, under the favour of their greater proximity to the sun, and which is reflected to us from a shorter distance.

§ 62. Seeking naturally in every celestial body for every one of the general phenomena which the earth presents to us, we expect also to find an atmosphere; that is, to find them surrounded by a fluid of such tenuity as to admit the passage of light, and therefore be perceptible to our senses only by its mechanical effect. The general principles of attraction acting upon all matter in the determination of the location of the bodies around its centre, according to their specific gravity, has caused a very marked distinction between the solid body of the planets and their atmosphere, such as we observe on the earth. While the first appears steadily balanced by the proportion between the cohesion and the gravitation of the parts, the density of the atmosphere diminishes with the distance to a degree as yet unknown, but which is probably limited by the equilibrium between its weight, that is gravitation, and its elasticity. So we expect to find upon the sun, as well as in

every planet, a part analogous to our atmosphere in its general constitution and properties, but approportioned, in respect to quantity, to the mass of the sun or planet; and, in respect to density, to the density of the mass of the sun or planet. The result in TABLE II. shows the density of the sun 0,29 or not $\frac{3}{10}$ of that of the earth, and its specific gravity, under the supposition that of the earth to be 4,7150 times that of water, will become, 1,37, or about similar to the resinous substances upon the earth. Atmospheric air having a specific gravity 0,0012 of the same scale, (or $\frac{32}{10000}$ of the water,) if we suppose the atmosphere of the sun of a density bearing the same ratio to that of its mass as that of the earth bears to its mass, we obtain for the density of the solar atmosphere, in the same scale, 0,00035 nearly; and, therefore, if of the same elevation as our atmosphere, it would support a column of mercury of nearly 8¾ inches, in the barometer, when the atmosphere of the earth would support 30 inches. This small specific gravity of the atmosphere of the sun must, besides, be under the influence of the high temperature of the sun, and thereby suffer an expansion much greater than what we can form an idea of, by comparison with our atmosphere. By the reaction of this, according to the general principles observed on the earth, that matter *absorbs* so much the more heat as it is less dense, we must again conclude, that the greater heat of the solid mass of the sun will be so much less sensible, both by its own lesser density, and by the tenuity of its atmosphere; as this has been proved to be the case in rising in the higher regions of our atmosphere. All these circumstances tend to a considerable equalization of the *sensible Temperature or Heat* upon the surface of the sun, to that of the earth, and probably even in the whole system in general. Thus, the question of the habitability of the sun and the planets for intellectual beings, is, by a simple consideration, brought within narrower limits than the knowledge possessed by an inhabitant of one part of the globe, in regard to the population of another, had attained but a few centuries since. Hence differences in organization

may exist between them and us, hardly greater than between us and some of the higher classes of animals on earth.

§ 63. If all these facts and their consequences have led us to find around the *sun* an atmosphere of great tenuity, the magnitude of its volume and mass must again lead us to infer that it must have *a great extent*, and that it must obey the effect of the rotation of the sun in a greater degree, and therefore show a *great ellipticity*. The denomination of *Zodiacal light* has been given to an appearance of a whitish light, which is seen after sunset in the spring, or before the rising of the sun in the fall, in the form of a section of a lens, within certain limits more or less inclined to the horizon, terminating almost in a point, at from 45° to 120° distance from it, and having evidently the sun in the middle of its broadest part, which is from 8° to 30° in breadth at the horizon; the longest axis of the eliptic or lenticular section is found to coincide with the equator of the sun; it is transparent, for we see the stars distinctly through it. The same causes which occasion the great light of the sun, give also to its atmosphere a greater light than ours, and, as we see every object only by the comparison with others, the greater light of the atmosphere of the sun becomes apparent to us when we are outside of it. These circumstances very soon caused the Zodiacal light to be considered as the atmosphere of the sun, to which it was evident that it belonged; the reasonings presented in the preceding sections, grounded upon the general principles of natural philosophy, as applied to the sun and the celestial bodies in general, all concur in confirming this explanation of the phenomenon; and the general theoretical principles of celestial mechanics show that such a form and tenuity must be found in the atmosphere of the sun.

A comparison of the angle of its extent, measured from the horizon at sunset, with the greatest elongation of Venus as seen from the earth, shows that this atmosphere extends nearly to the orbit of Venus even in its smallest extent; and as it sometimes exceeds a right angle, we must

conclude that it may, under certain circumstances, exceed the distance of the orbit of the earth from the sun. The same reasons which have accounted for the great extent and tenuity of the solar atmosphere, evidently cause a great variability in its extent, as we observe in the Zodiacal light; because every variation which the combination of the natural powers produces, has so much a greater effect, as the magnitude of the atmosphere is more extensive proportionably to its density. And if such an extension occurs when the earth is in the protraction of the nodes of the equator of the sun with the ecliptic, and at the same time in the greatest proximity to it, the earth falls itself within this atmosphere; it becomes, therefore, invisible to us. The most favourable times for observing it are evidently when the earth in its orbit is at right angles to the nodes of the solar equator, which is when it is in 5ˢ 10°, that is, in the spring; then the Zodiacal light is best visible in the evening following the sun, and in 11ˢ 10°, in the fall, when it precedes the rising of the sun. The variability of its extent and breadth is evidently a consequence of the position of the earth in its orbit in respect to the equator of the sun, and its nodes, and of the variability of the physical state of this atmosphere itself. The mean extent of this atmosphere is represented in PLATE I. and II. by the dotted line which is seen in them.

§ 64. Rays of light passing through air of different density will deviate from a straight line, in such a manner as to bend towards the more dense medium, in a certain proportion to the density of this medium. The atmosphere presents a medium regularly increasing in density from the higher regions to the surface of the earth; it produces, therefore, the phenomena of a constant deviation of the rays of light, in a direction perpendicular to the horizon: this phenomenon is called *Refraction*, and the quantity of the angular deviation of the rays of light observed at the horizon is called the *Horizontal Refraction*, which is the greatest of all, because the passage

of the ray of light through the atmosphere is in that case the longest; it amounts on earth to about thirty-three minutes in the vertical arc. This angle of refraction is of course subject to variations proportional to the changes of density of the atmosphere; of course principally to those indicated by the barometer and thermometer, augmenting with the elevation of the barometer, and diminishing with the augmentation of the degree of the thermometer.

Furnished with this induction in respect to the nature of an atmosphere, by observations upon our own, we naturally inquire also, in every other atmosphere, after the same phenomenon. As we have found for the sun an atmosphere supporting only an elevation of the barometer of 8⅔ inches, under the same height at which our terrestrial atmosphere would support thirty inches, and with a temperature so much superior to any that can take place upon earth, as to exceed near 40,000 times the degree of incandescence, we see that the refraction of the solar atmosphere must be reduced to a quantity so minute, as to become imperceptible for our means of observation, as a simulated calculation, under the above supposition, would easily show. We, therefore, cannot ascertain the existence, or any of the properties of the solar atmosphere, by means of refraction, which is, what we might call, a fractional effect of it.

§ 65. Let us here anticipate for a moment the knowledge which we shall only derive from future inquiries into the individual nature of the planets. All planets unite to their revolution around the sun a rotatory motion around their axes; this last we have found in the sun. The principles of mechanics show this rotation to be a consequence of the revolution around a point of attraction; we may be authorised to reverse this conclusion, and infer that the rotatory motion of the sun, indicates a revolution around a centre of attraction, accompanied by all the planets which revolve around it, in the same way as the planets make their revolution around the sun accompanied by the satellites. After having seen,

by the size of the orbit of Uranus, the immense distance to which the attraction of the sun can extend, and, by the time of its revolution, compared with that of Mercury, or even of the satellites of the planets, the great difference of duration, which the epochs of the revolution of the celestial bodies present, we must ascribe to the orbit in which the sun revolves a diameter of such immensity, and a time of revolution of such a length, that the whole series of ages to which our history extends is too short a time to permit its progress to have been observed, or to enable us to determ'ne its direction. From the comparison of the diameters of the orbits of satellites, hardly expressible in fractions of the diameters of the larger planetary orbits, and the times of their revolutions about their primaries, varying between 22 hours and 107 days, compared with the revolution of Uranus, of 84 years, we have an ample field for conjecture upon the next steps in the scale, from the dimensions and times observed in the solar system, to those in which the sun is itself acting only a secondary part.

Several astronomers have attempted to determine the direction of this orbit of the sun, from what is now called the proper motion of the fixed stars, for we must of course look for the effect of this motion in the appearances of objects external to the solar system, and we have learned by the inversion of the appearances of the planetary motions viewed from the earth to the motion of the earth itself, the happy result of such an inverse mode of proceeding in the investigation of astronomic truths. The direction of the motion of the sun in its orbit, since the time that more accurate observations can furnish data for conclusion, was thought by Herschel to be towards the star marked δ in the constellation of Hercules; but other astronomers found results contradictory to it, and found other directions probable. It is reserved for the future exertions of astronomers, for greater perfection in the means, and a longer period of accurate observations, to give to the science the data and the degree of elevation which will admit the dec sion of this and similar questions.

CHAPTER II.

Of the Planets between the Sun and the Earth.

§ 66. We are deprived of the advantage of observing *Mercury*, the planet nearest to the sun, during so great a part of his revolution, by the superior brightness of the latter, that our knowledge of his individual nature is but scanty. His greatest apparent diameter is 11″, 37, and his smallest 6″, 3 while the earth must appear to the observer in Mercury, in these same situations, respectively, under an angle of nearly 28″ and 12″, 5; this, as has been before shown, is equal to twice the parallax of Mercury. Seen from the sun, whence it is distant only 42 solar diameters, it must appear under an angle of only 18″, 38 at a mean, while the sun must appear to it under an angle of 1° 21′ 28″, or nearly three times as large as to us. The rotation of mercury upon his axis, has been determined to be 24^h 5^m 20^s, and his elipticity $\frac{1}{126}$ part of the diameter, in consequence of his velocity of rotation; but as he presents us no well determined point upon his surface, and his appearance may be considered as almost momentary, for an observation which requires so much minuteness, (for the proximity of the sun makes detailed observations of this planet impossible in the day time, with our optical powers,) these data cannot be obtained in a high degree of accuracy.

The real diameter of Mercury, deducible from these data, is less than one-third of that of the earth, its surface only about one-seventh, and its volume only about $\frac{1}{22}$; comparing these

fractional quantities with their corresponding magnitudes in the sun, we obtain the minute fractions which are found in the column for Mercury, in the lines 6, 8, and 10 of Table II. The mass of this planet can be determined only by its influence in the deviation of Venus and the earth from exact eliptic orbits, or what are called the perturbations, which are besides but small; as it has no satellite, the most direct road to this determination cannot be taken. This mass has as yet been rather determined by supposition, directed in part by an observation of Laplace : that the density of the planets nearest to the sun is the greatest, and by a reasoning somewhat analogous to that we have pursued, in relation to the heat of the sun and planets, it is estimated at only about $\frac{1}{8}$ of that of the earth, or $\frac{5}{10,000,000}$ of the mass of the sun; it presents us, therefore, with a density nearly three times as great as that of the earth, or twelve times that of the sun, (as seen in Table II.) This density, acccompanying so small a diameter, occasions the fall of heavy bodies upon its surface to be even more rapid than upon the earth, for, from these data, it will amount to 17,77 feet in the first second of time.

This great density of mercury would also lead us to infer an atmosphere of greater density and refractive power than that of the earth, but of small altitude, on account of the small mass of the planet; the proximity and great light of the sun will not allow this point to be determined by observation.

§ 67. Mercury must present to the earth in his different situations in his orbit, in respect to the place of the sun and that of the earth, in its own orbit, a variety of appearances; these have relation to the part of the planet which we see illuminated, and that, which, being turned from the sun, escapes our distinct vision. These different appearances, form what is called the phases of a celestial body; of this phenomenon we have the most striking instance in the phases

of the moon, but under different circumstances, as will be shown in treating of the moon. While, in that case, the spectator is placed within the orbit of the body receiving its light from the one without it, the phases which the inferior planets present to us are such as are seen by a spectator outside of the orbit of the body revolving around the centre of light, as is represented in PLATE V. fig. 1; in the instance both of Mercury and Venus, in their actual proportional distance compared with that of the earth, the phenomena are exactly similar with those of a round body turning around a light before the eye of an observer placed without the circle of its revolution.

If we consider a triangle to be formed between the Sun, the Earth, and Mercury, the angle of this last which we see illuminated, will always be equal to the sum of the angles at the sun and at the earth, in this triangle. We see Mercury entirely full, only when he is directly behind the sun. When in such a situation that the angle at Mercury between the sun and earth is a right angle; that is, in its greatest elongation or digression, we see it like what we call a half moon, and we of course lose sight of him entirely when he is in, or even only near, the line between the earth and sun, unless the respective position of the earth and Mercury, in relation to the nodes of their orbits be such as to occasion Mercury to appear projected upon the disk of the sun, which constitutes what we have already spoken of as a *transit of Mercury*. In this situation Mercury appears as a dark circular spot upon the sun, and therefore exhibits his full diameter. We have in this situation the only opportunity of determining his diameter with accuracy.

§ 68. *Venus*, the second planet between the earth and the sun, and the nearest to us of all, is much more interesting to us than Mercury, for we can observe her easily, being less immersed in the light of the sun. The difference between the extremes of her apparent diameter, is of course so much the greater as her orbit is greater, and her distance from us

varies the whole of its diameter. When nearest to us, and in the transit over the sun, Venus appears about six times larger than when at the greatest distance on the other side of the sun; in this last case it appears to us under a diameter of only 9″,6, while in the former it subtends an angle of a whole minute within two-tenths of a second. At a distance equal to that of the sun, it appears under an angle of 16″,8; viewed from the sun, it must appear at a mean under an angle of $22\frac{1}{2}$ seconds, and the sun must appear to it under a mean diameter of about $44\frac{1}{3}$ minutes. The angle which the earth subtends to an observer in Venus, corresponding to the two extreme distances, the half of which forms for us the parallax of this planet, is, at her greatest proximity, 61″,4, and at her greatest distance only 10″. The comparison of these relative appearances shows Venus to have a magnitude not much different from that of the earth. The result of observations, repeated, and made with great care, give for the actual lineal diameter of Venus 0,9593 of that of the earth, or taking the sun as the unit, only the 0,0086 part of its diameter. The surface and other dimensions, consequent and corresponding to this element, are recorded in their respective places in TABLE II., to which reference may be made. The inferior conjunction of Venus must of course always be invisible to us, because the planet presenting the dark side towards us, is invisible; the superior conjunction, where she is in full light, becomes visible only when the latitude is as much, or more than one degree, as, by a greater proximity to the sun, his intense light prevents the planet from being visible.

§ 69. The mass of Venus is to be determined principally by the effect of the mutual attraction between her and the earth, or their perturbations, the absence of a satellite depriving of the power of employing more direct and more accurate means. By the latest calculations, it has been assumed at 1,2118; this is about two-tenths more than the mass of the earth; it had before been assumed at 0,9243 or somewhat less than the mass of the earth; both suppositions

give the density of Venus greater than that of the earth; namely: the first, 1,48, and the latter, 1,3726; in like manner, also, we obtain for her specific gravity 6,472, or 6,615; these results, though apparently discordant, are, however, sufficiently accurate to be used in the cases to which they are principally applied in astronomy, namely: to determine their effect in the perturbations. It has been attempted to determine the horizontal refraction of this planet by its effect upon the twilight, by which it was determined to be about $30\frac{1}{2}'$ in the horizon; but it is easy to perceive that this must rather be a minimum, because the exact line that separates light from darkness must escape the acuteness of observation at a distance equal to that of one planet from another.

Spots discovered upon the disk of Venus have furnished the means of determining her rotation upon an axis inclined about 15 degrees to the plane of the orbit of the planet, and which is performed in the time of $23^h\ 21^m\ 19^s$, or only $34\frac{3}{4}$ minutes less than the rotation of the earth upon its axis. The resulting eccentricity of $\frac{1}{310}$ is also very near that of the earth; and the fall of heavy bodies upon its surface must be 16,16 feet in the first second of time. Certain minute observations indicate mountains upon Venus of more than 140000 feet high, or about $\frac{1}{144}$ of the diameter, therefore much greater than are found upon the earth. Though the greater density of the planet might authorise the supposition that its parts would come together (or agglomerate) under greater irregularities than those of the earth, still, these observations, which are deduced from the length of the shadow of the mountains, (the direct measurement being too minute, if even possible,) are not entitled to any high character for accuracy, as may well be imagined in a determination that depends upon such minute observations. These kind of facts, therefore, are never advanced with a degree of confidence in any manner comparable to that with which the revolutions of the planets in their orbits and the greater celestial phenomena are stated.

of late within small limits of variation, the most accurate results falling between $\frac{1}{309}$ and $\frac{1}{310}$ of the diameter, though some results came as low as $\frac{1}{332}$ or up to $\frac{1}{293}$, $\frac{1}{289}$, according to the methods or places of observation made use of. The figure of the earth being necessarily the result of the equilibrium between the gravitation of the matter of the earth, and the cohesion, or its resistance to a change of form, some inequalities may be expected in the results deduced from the comparison of different places upon the earth, which close inquiries may discover.

By reason of this elipticity, we find ourselves, on different points upon the surface of the earth, at different distances from the centre; we experience, therefore, in our movement upon the surface of the earth, in relation to the appearances of the celestial phenomena that occur at a distance sufficiently small to make the difference between the diameters of the earth sensible, an effect exactly analogous to that which we have spoken of before, as arising from the circumstance of the eliptic form of the orbit of the earth; namely, those appearances will all be affected by an inequality within certain limits, proportionable to the difference of the radii of the earth; this will be the most observable in all the phenomena which relate to the moon, our satellite, because it is the heavenly body nearest us.

The specific gravity of the earth has been determined by ascertaining the deflection of a plumb line from the vertical, under the action of the attraction of a mountain, whose specific gravity appeared to be capable of being ascertained.

§ 72. The phenomena dependent on the rotation of the earth upon its axis are of the utmost importance to us; the appearances of all objects situated without the earth are naturally affected by it, and we are by that rotation placed in the position of a man, who, in performing a walk around an extensive area, would constantly turn round upon himself. By this compound movement, the whole spectacle of the universe appears every day to circulate round the earth, in a

direction contrary to that which we have hitherto found common to all the planets, and the gradual daily changes in the appearances, which result from the greater motions of the heavenly bodies, appear to assume proportionally a minor importance.

As the sun, around which the earth performs its revolution, must necessarily appear to us at any time, in a direction diametrically opposite to that which would be assigned to the earth in the preceding description of the courses of the planets in their orbit, astronomical tables, keeping up the language of appearances, even after we are better informed, are calculated to represent this apparent motion of the sun.

All the heavenly bodies appear daily to emerge from the eastern, and sink again from our view beneath the western horizon, after a shorter or longer stay. This we find variable,* according to their position in an arc supposed to be drawn perpendicular to this apparent course, and according to the place of the observer on the earth's surface; the observer at the earth's equator will see all the heavenly bodies, without exception, performing these motions uniformly and regularly, in directions perpendicular to the horizon, because for him the axis of rotation lies in the horizon. The two opposite points of the axis of rotation thus perpendicular to the equator, form what are called the poles of the earth, as well when applied indiscriminately to the direction of this axis, as when applied individually to the points on the earth equi-distant from the equator on all sides. As the observer approaches more towards the one or the other pole, some of the stars near to the pole towards which he moves will remain above his horizon during the whole time of the revolution of the earth, or they will perform their whole revolution without setting to him; while some of those that lie near the poles opposite, will disappear entirely, and therefore perform their revolution under his horizon; this disparity between the appearances of the celestial bodies, arising from their position in respect to the poles of the earth, will increase in proportion

as the observer approaches to the pole, in which, if it could be attained, he would see only just half the spectacle of the heavens.

§ 73. The rotation of the earth upon its axis, in absolute space, or in relation to the infinite distance of the fixed stars, constitutes what is called in astronomy, the sideral day; it is usually divided into twenty-four equal parts called sideral hours; this simple statement shows that all the phenomena depending on the rotation of the earth upon its axis, and all motions considered as parallel to it, or inquired into, in relation to this motion, that is parallel to the equator of the earth, will naturally be counted or measured by this astronomical time.

But during the time of such a rotation of the earth upon its axis, absolutely speaking, it has also advanced in its orbit a certain determined distance; and until a certain determined point of the surface of the earth has again arrived in the plane perpendicular to the daily motion, and passing through the sun, from which it had appeared to start the preceding day, a certain part of the revolution of the earth in its orbit must still be performed, which is again measured by the time employed in performing it; this time is, at a mean rate, 3′ 56″,555 of the same sideral time.

If, therefore, we adopt and divide the year as it appears to us by the number of returns of the sun, during the time that the earth performs a whole revolution in its orbit, we obtain, omitting smaller changes, a *mean solar year*, and a *mean solar day*, for each apparent return of the sun to the plane above mentioned, which is called the meridian of the place; such a *Solar day* is equal to 24^h 3^m 56^s,555 sideral time.

As all our occupations in life must be regulated by the presence of the sun, the day, as commonly understood, is regulated by the time just mentioned, which constitutes what in astronomy is called *a Mean solar day*, and this is again divided into twenty-four parts, or mean solar hours; mean solar hours are therefore longer than sideral hours, and a sideral day has only 23^h 56^m 04^s,0907 of mean solar time.

By the law of the revolution of the planets in their eliptic orbits, explained in the beginning, we know that they describe sectors of their orbit, the surfaces of which are equal in equal times, or as it is usually expressed, proportional to the time; the distance of the earth from the sun being variable, this law of equal areas in equal times necessarily requires that the arcs described in the equal times, be inversely as these radii, and consequently unequal; therefore these successive returns of any one point of the surface of the earth to the plane called the meridian, passing through the earth's centre this point and the sun, and perpendicular to the equator, must be unequal also.

These intervals between the actual passages of the sun through the meridian, to use the ordinary language descriptive of the appearances, are therefore unequal; and from this results a third kind of time, namely, *True time*, because it indicates the true position of the given point on the earth, in respect to the sun, which is the result of the observation of the sun; it is also called *Apparent time*, and, in fact, with more propriety, because time being a measure of duration, which like all measures ought to be uniform, it follows that the most proper way of expressing this result of the unequal motion of the earth in its orbit, is to indicate for every day the mean time at which the sun will (apparently) pass the meridian; the difference between these two times is called the *Equation of time*.

Either this Equation, or an indication of the *Mean time of True mid-day*, that is, the mean time of the passage of the sun through the meridian, are always given for each day of the year in astronomical almanacs. The difference between true and mean solar time may amount to as much as $16^m\ 16^s$, and the daily variation of this difference from one day to another may amount in the extreme to thirty seconds.

These are the principles of the three kinds of time which it is necessary to use, according to the peculiar circumstances of the application.

It will be at once seen, that true solar time being irregular, is not susceptible of being represented by a mechanical arrangement or machine, *a Clock* for instance, of such simplicity as is desirable for a high degree of accuracy; only *sideral* and *mean solar time*, are therefore usually pointed out by clocks; the first is exclusively reserved for observatories, because, as we have above observed, it contains the direct measure of any movement or arc on the equator.

Mean solar time is often, and even generally, more convenient for an observer not stationary in the same place, and ought to be in general the regulator of the *Public clocks*, upon which the affairs of common intercourse and daily occupations depend.

§ 74. The inclination of the plane of the equator of the earth to the plane of the ecliptic, is, in the ordinary language which represents mere appearances, called the obliquity of the ecliptic. At the beginning of the present century, this angle was 23° 27′ 57″. It is subject to periodical changes or oscillations, which principally depend upon the position of the orbit of the moon, with whose periodic motions these changes correspond, and which leave as a mean constant result, a diminution of 0″, 52, or about half a second each year since the earliest astronomical records. The general laws of gravitation prove that this diminution will change into an augmentation—that it therefore is of an oscillatory nature; but the period of this oscillation is of many thousand years, and too long to be ascertainable from observations in the comparatively short time to which those that can be depended upon extend.

It is evident that every point of the surface of the earth must have a motion parallel to the equator, in partaking of the general daily rotation of the earth; a motion which the observer at that point refers back to the celestial bodies themselves, where it produces a new variety and an additional complexity of the appearances under which the real motions of the heavenly bodies must appear to the individual ob-

server. The apparent motion of the planets, as affected by the eccentric position and proper motion of the earth, by which they appear to advance or retrograde alternately, in planes differently inclined to the ecliptic, receive from it again a different inclination of their apparent oscillatory course. The first general remark which might strike us as a consequence of this individual motion of the observer, is a general inversion of all the motions, by which they pass every twenty-four hours before the observer, in a direction and order the reverse of that in which they would really appear, if the earth had no rotation upon its axis. From this apparent effect, therefore, every observation is first to be freed, in order to determine the apparent position upon the supposition of the earth's being at rest; and from the eccentric position of this, its real position in its orbit is then to be deduced, as it refers to the sun as a central body.

§ 75. The near approach of the obliquity of the ecliptic to constancy, within the limits of the periodical oscillation above mentioned, indicates that the axis of the earth remains parallel to itself during the time of the revolution of the earth around the sun—that, therefore, it is not essentially affected by this revolution. In consequence of this circumstance, the plane of the equator must, in the course of the yearly revolution around the earth, present to the sun its two sides, with their corresponding poles alternately; and therefore pass twice through that position, where a line drawn to the sun lies in the protraction of this plane. This circumstance is the cause of the phenomena of the changes of the seasons, the variation in the length of the day, and the several consequences which affect what we might call the domestic concerns of our earth, considered as the habitation of men and animals, and the supporter of vegetation; it thus gives rise to the most striking and most extensive subdivisions of the earth; as also to the principal great subdivisions of the year. Merely to state this will be sufficient here; a full description of the consequences may with more propriety be deferred to

the more detailed account of the earth, to which we shall devote a separate chapter.

§ 76. The *Earth*, considered *as a planet*, presents an example of a system of celestial bodies revolving around their common centre of gravity, in the simplest possible form; there being only two bodies concerned, namely, the *Earth* and its satellite, the *Moon;* this satellite makes its revolution around the earth at a mean distance of about thirty diameters of the earth, and performs nearly thirteen revolutions around the earth during one single revolution of the earth in its own orbit. In this system, therefore, we have an opportunity of studying the principles, and shewing the effects of the mutual attraction of the celestial bodies in their simplest form.

But although the case is the simplest, the proximity of the moon renders all the effects of their mutual attraction the most sensible to us, and we see, in the most distinct manner, all the individual influences which remain imperceptible to us in the more distant celestial bodies. The moon is nearly, if not quite, the smallest of all the planets and satellites of our solar system; still it appears to us under an apparent diameter equal to that of the sun itself; a circumstance which is probably not the case with any of the satellites of the other planets; the influences it receives from the sun and the earth, bodies so vastly superior, occasion of course a combined variation of its course, so much greater and more sensible as they are exercised upon a smaller mass; and they are observed by us under greater angles than any similar influence upon other celestial bodies can present themselves to us; for astronomy presents us only angular distances; these variations acquire, besides, an importance so much the greater as the motion of the moon has a peculiar interest for us.

Still more extensively important, though smaller in their angular magnitudes are the influences of the moon upon the position of the earth, which suffering small derangements, places it in a constantly changeable relation to all external

objects, which are for this reason affected by a variable, and in the sense in which all these changes are considered, oscillatory motion, of which it becomes necessary to keep an account in the results of all observations.

In this treatise no more can be done than to develope the principle of these influences from the primary idea of the law of attraction. A clear representation of what must be the consequence of it in the case, is all that can be done without mathematical investigations, which do not come within the limits of this work.

§ 77. In the last column of TABLES I. and II. are collected the data relating to the MOON corresponding to those given of the planets; without repeating them, we will only make use of the principal results and their consequences.

The orbit of the moon is elliptic, the earth approximately occupying one of the foci of the ellipse, so that its greatest distance, or the apogee, is 32,744 diameters of the earth, and the perigee, or nearest distance 27,958 diameters. We say approximately, because, by the principle of universal gravitation, any two celestial bodies revolving in connection under its influence, must perform revolutions around their common centre of gravity, proportional in magnitude inversely to their masses, so that the centre of any one of them could be the centre of motion of the other, only under the supposition of the other being infinitely small, a case only possible by mathematical supposition, and not to be met with exactly in material nature.

The inclination of its orbit to the ecliptic was, about 5° 08′ 38″ at the beginning of this century; and this inclination combines with that of the equator of the earth, to the ecliptic, adds to, or subtracts from the apparent deviation of the sun from the equator, so as to produce a very variable apparent motion of the moon. This variation is so much the greater as the elements of the lunar orbit are all variable within short periods, which gives to the moon the appearance of a winding motion in relation to the two planes of the ecliptic and the equator.

The perigee of the moon makes an entire revolution of the whole orbit in 3231$\frac{1}{3}$ days. The nodes perform one complete revolution in about 6793,4 days. The variations of the latitude of the moon are of course affected by these changes, and they furnish a variety of points of comparison for the revolution of the moon in its orbit, according to the point to which the revolution is referred, either in its own orbit, in relation to the sun, or, absolutely speaking, in respect to the fixed stars or to our artificial division of the ecliptic. The following are the times in which each of these revolutions are performed at a mean rate, viz:

	days.
The *moon* returns to the same *Node* in - - - - -	27,212222
" " " Apogee, or performs what is called an anomalistic revolution, in	27,5546
" " " Equinoctial point in - - -	27,321582
" " " Sun apparently, or performs a synodic revolution, in - - - -	29,5306
" Performs a revolution, absolutely speaking, in - -	27,3216

It is evident that these times must be affected by all the influences of the mutual attractions of the sun, the earth, and the moon, the effects of which are variable.

The apparent angular motion of the moon, in its orbit, is in a mean 13° 10′ 35″ each day. It therefore presents to us the greatest angular motion of all celestial bodies, though it has, absolutely speaking, the smallest velocity, as may be seen by the comparison of the data of line 10, TABLE I.

§ 78. This rapid change of appearances, and the various complication of the cases, rendered the most minute researches and most accurate observations necessary, before the place of the moon could be determined for a given time by previous calculation. Without a theory brought to a high degree of perfection, it was impossible to account for variations so complicated as to approach almost to the arbitrary. To the high degree of interest, which the desire to overcome these difficulties must awake, was joined one closely connected with the wants, safety, and enjoyment of man.

We have already seen that time is measured by the rotation of the earth around its axis; in consequence of which, each point upon the surface of the earth will count different hours, or subdivisions of the day, at any absolute moment of time; in counting from the moment when its own radius has been in the plane passing through the centres of the sun and the earth, and perpendicular to the equator. This we have seen to be the point of departure for the time of each place, or what is called in common language mid-day. We have also seen that all celestial motions, and consequently their differences, were measured by this unit (the day) and its subdivisions. The very sensible difference between the apparent angular velocity of the moon and those of the sun or planets, or its comparison with the position of the fixed stars, naturally suggested the idea: that, under the supposition of an accurate knowledge of the motion of the moon, the very time which would elapse between the passage of two places upon the surface of the earth through the plane passing through the centres of both, could be measured, if for a certain distance of the moon from the sun or a fixed star, the times of these two places were known. The time of such distance denoting an absolute moment, the difference of time counted at the two places, is of course equal to their different situation in the rotation of the earth, which is called their difference of meridian, or longitude. Or, inverting the case, if in two places, whose meridians are determined by fixed observatories, the transit of the moon through these meridians is observed, the differences of the places of the moon in its orbit, corresponding to these transits, will, by the present accuracy of our knowledge of the moon's motion, allow us to determine the time which the moon will employ to describe the intermediate space, and this time will again be the difference of longitude of the two observatories. In this, consisted the solution of the problem of longitude, as it is usually called, and which has been so extremely valuable for the security of navigation. In this case, therefore, the demands of ordinary

life have subserved the interests of science, from which they have in return received a complete solution of the problem in question.

§ 79. By the principles already quoted, the apparent diameter of the moon must necessarily undergo changes, inversely proportioned to its distance from the earth; we find it therefore varying between 29′ 30″ and 33′ 30″, at its greatest and smallest distance from the earth; these, by comparison, will be found to be alternately more or less than the apparent diameter of the sun. From the same circumstance, also, the raidus of the earth must subtend an angle at the moon varying in correspondence with this distance; that is, the parallax of the moon must vary in a ratio inversely proportional to the distance, and it is found to vary between 61′ 26″ and 53′ 46″. Both these elements are therefore changing as rapidly as the other elements of the lunar motions, or appearances, and in astronomical calculations it becomes necessary to pay regard to their existing temporary magnitude.

For what relates to the mass of the moon, and the other data that depend upon it, the reader may consult the tables; and it may be more proper here to treat of the principles and consequences of the mutual connections between the earth and the moon, which are extensive, important, and of general influence upon the individual astronomy of the earth.

§ 80. The attraction of the moon upon the earth acts from an orbit having a position itself subject to a rapid revolution, the inclination of which, towards the equator, is variable; the earth, upon which this attraction is exercised, is not an exact sphere, but an ellipsoid, that is, flattened at the poles, and elevated at the equator; it therefore presents to this attraction what might be considered an overhanging mass, exposed more favourably to this attraction, and in a position variously inclined to the variable orbit of the moon; the effect of the attraction is therefore similar to the effect of a weight attached to the outside of a sphere, in all directions successively, and always perpendicular to the plane upon

which it lies, that is to the planes of the equator. Hence is occasioned, in every position, a leaning of the axis towards the side of the weight, represented by these combined effects, by which the pole would describe a small circle in the course of a lunar revolution, considered circular. But the variable distance of the moon, and consequent proportional variation of the power of this attraction, acting, as we have seen in all cases, so as to produce an elliptic motion, the *axis* of the earth actually describes a small *ellipse*, changeable in consequence of, and in correspondence with, the change of position of the lunar orbit; so that we find the position of even the axis of the earth is not constant or invariable.

This motion of the axis is called the *Lunar Nutation;* the ellipse thus described by the pole has for its greater axis 23″,7263, and for the smaller axis, 9″,643 of a great circle; the direction of these axes of the ellipse evidently follows the change of position of the orbit of the moon; and as this motion is measured by, or the same as, that of the nodes, the position of the ascending node of the moon's orbit in the ecliptic, is the leading element by which the influence of the nutation is calculated for any given time or point of the celestial sphere. Such an element of a variation is called, in astronomy, its *argument.* In this case, as in all others, we make use of a language conformable to the appearance produced by this motion, and transfer our own motion, by inversion, to the whole external spectacle of the heavens.

§ 81. The principles here explained: shew that the inclination of the plane of the equator of the earth to the ecliptic must subject the earth to a similar oscillation, arising from the attraction of the sun upon the prominent ellipsoidic parts of the earth's equator. This effect must, however, be a great deal less sensible, on account of the great distance of the sun occasioning the angle under which this influence is exercised to be very small. This effect is called the *Solar Nutation;* the corresponding great and small axes of the el-

lipse are only $0'',6$ and $0'',4$; the smallness of this effect renders it sensible only in the direction perpendicular to the earth's orbit, that is in the latitudes; or perpendicular to the equator, that is in the declinations; the projection of it upon the planes of the ecliptic or the equator become too small to be perceptible to our means of observation, and are, therefore, not taken into account. The epoch of the changes of these influences depends evidently on the position of the sun in respect to planes of the equator, hence the distance of the sun from the node of the equator, that is his longitude, forms what we have called in the preceding section, the argument of this *solar nutation.* Of course, also, this small motion is transferred from its reality to the apparent effect which it presents to us; that is, we correct (as this is termed in astronomy) the position of all heavenly bodies for this effect, to register their position independently of the temporary inclination of the earth's axis, whether it arise from the solar or lunar nutation.

§ 82. We are now prepared to find mutual influences combining to produce motions more complicated than were presented at the first outset, though still consequences of the same law of universal gravitation. The connexion of the moon with the earth, in her revolution around the sun, is the cause that what we have said of the earth's orbit and revolution around the sun, does not properly refer to the earth's centre, but to the centre of gravity of the combined mass of the earth and the moon.

By the principles of universal gravitation, these two bodies, and, in general, any two celestial bodies similarly connected, perform revolutions around their common centre of gravity in ellipses, the axes of which are inversely proportional to their masses, or directly, as the distances of their centre from the common centre of gravity.

Thus, therefore, for this reason also, the earth performs her course in a kind of undulating and unequal manner, the effect of which will again fall upon the appearances of all

celestial bodies as seen from the earth, and occasion a correction dependant on the position of the moon. This effect, joined to a similar one resulting from the planets, may oscillate to 1″ on each side; therefore produce, in the whole, a difference of about 2″ in the latitude of the earth.

The common center of gravity of the earth and moon lies within the earth; this is likewise proved by the inequality which this produces in the revolution of the earth around the sun, the maximum of which is 7″,5, while the parallax of the sun, that is the angle subtended by the radius of the earth at the sun, is 8″,7, and therefore greater than this variation; the earth is, therefore, from that cause, never in its extreme, displaced as much as the whole of its own radius.

§ 83. The moon being thus, as we may say, attached to the earth in its revolution around the sun, and this revolution itself being always influenced by the mutual action of universal gravitation upon the united mass, must naturally be affected by the changeable position of that part of it which constitutes what are called the *Perturbations* of the earth by the moon; into the numerical details of these we do not intend to enter in this work; they belong to a higher department of astronomy. The point from which the effects of these perturbations of the motions of the earth in its orbit by the influence of the moon, may most naturally be counted periodically; that is, what we have above called the argument of these perturbations, may most naturally be placed in one of the nodes of the moon's orbit with the ecliptic, as on this the situation of the lunar orbit depends; and by which, besides the place of the moon in its orbit, the situation of the line of attraction of the sun and the moon, in relation to each other, is determined.

§ 84. Finding the influence of the moon upon the revolution of the earth already so remarkable, notwithstanding the small mass of the moon, we are naturally prepared to see the effect of the sun upon the moon, in its revolution around the earth, so much greater, and so much more varied. Though

the moon, by performing its revolution around the earth, shews: that the preponderance of its attraction over that of the sun is due rather to its proximity than to its mass; the sun, which exercises a far more preponderating power over so many planets, at so much greater distances, and of so much more mass, must exercise a decided influence upon all the details of the motion of the moon, in conformity to the general principles of attraction; and all the effects must be so much the more sensible to us, as the moon is the celestial body nearest to us; her movements, in consequence, subtend the greatest angles, proportionally to their real magnitude; or, simply, the *perturbations* of the moon must be the greatest, and the most varied that we can observe; therefore, we observe even the changes of the elements of the orbit itself, produced by them, and the effect of the combinations of the situations, and the varied coincidence of the cases.

Thus it happens, that, in the present highly advanced state of the science, the tables of the lunar motions require the application of *thirty-six equations*, (as the results of these variations of the mean positions, on account of these various influences, are called in astronomy,) for the accurate determination of the place in which we shall see the moon at a certain moment, and from a certain point of the surface of the earth; for it may easily be inferred from what has been stated: that, in this case, the local situation of the observer upon the surface of the earth, in comparison with the appearance that the phenomena relating to the moon would present from the centre, is also of greater apparent influence than we find in any other part of astronomy.

The deduction of all these details upon mathematical principles, is evidently the task of a highly cultivated analysis; and our task here can be no more than to show the fundamental effects of the solar attraction from which they are derived.

§ 85. The *first* and most remarkable effect of this attraction of the sun, exerted from a point *extraneous* to the lunar

orbit, will at once be perceived to be necessarily, a *Variation of its distance from the earth;* varying according as the moon, by its position in relation to the sun and the earth, is either under their combined attraction, or under the influence of only their difference. The first must be the case when the moon is in a position directly opposite to the sun, when it receives the attraction of both from the same side; or rather, is more abandoned to the attraction of the earth alone, and the distance of the moon is thereby made smaller, under the modification which we shall see hereafter; the second takes place when the moon being, in relation to the earth, on the same side as the sun, therefore, between sun and earth, and the attraction of the two bodies acts in opposite direction; consequently, as the sun draws the moon away from the earth, its distance from the earth is augmented. This effect of the sun's attraction leading the moon entirely out of the course which it would otherwise take, is called the *Evection,* the maximum of which is at present 1° 30′ 29″,9.

It is, in general, evident that the gravity of the moon towards the earth is diminished by the effect of the sun's attraction; in consequence of which the moon must describe around the earth an ellipse of greater dimensions than it would describe under the influence of the attraction of the earth alone; this augmentation of the distance is $\frac{1}{358}$ of the radius of the orbit; so much, therefore, the gravitation of the moon towards the earth is diminished by the sun's attraction. The angular velocity diminishes $\frac{1}{179}$, as the areas of the sectors must be the same for equal times in both orbits.

In the two syzigies the gravitation of the moon towards the earth is diminished nearly in the proportion of the radius of the orbit of the moon to that of the orbit of the earth. When, on the contrary, the moon is seen in a direction at right angles with the sun, this diminution is about half of that in the syzigies.

Between these extreme cases, all the intermediate changes, which correspond to intermediate positions of the moon,

in passing from the one to the other extreme, must evidently take place; these depend, of course, upon the angle between the line of attraction of the sun and the radius of the moon's orbit, which are moreover in different planes, and under a constantly varying inclination to each other. When these two lines are perpendicular to each other, this effect must vanish, and the same influence, which acted to prolong the radius of the moon's orbit, will, as we shall see, produce the maximum of an effect of a different kind, in two points diametrically opposite to each other in the lunar orbit; this will in its turn diminish when the evection augments; so that these two effects are always apparent, and combined in inverse ratio.

§ 86. This second effect is called the *Variation*, because it consists in a variation of the velocity of the motion of the moon in its orbit; it is evident that when the radius of the moon's orbit is perpendicular to the line of attraction of the sun, the whole effect of the sun's attraction is exercised in the direction of a tangent to the orbit at that place; it therefore acts entirely in acceleration or retardation of the moon's motion in its orbit; and this place of the moon's orbit is very near to that in which we see it at right angle with the sun, or what is called *in quadrature*, because the radius of the orbit of the moon subtends at the sun, an angle always less than $4\frac{1}{2}$ minutes, which, therefore, is the utmost limit of the difference between these two angles.

When the direction of the moon's motion is *towards the sun*, that is, during her motion between full moon and new moon, the variation is *in acceleration* of the motion, because the attraction of the sun acts in coincidence with the proper motion of the moon; when on the contrary the moon's motion is apparently *receding from the sun*, that is, from new moon to full moon, the attraction of the sun acting in an opposite direction to the moon's motion, will of course *retard it*. From this it is also easy to conclude that, as in the

evection, the smallest effect, or variation, takes place in the two points of the orbit at right angles with the greatest, which is, in this case, in the syzigies.

§ 87. These effects of the solar attraction are of course influenced, in regard to their magnitude, by the greater or smaller distance of the earth from the sun; or what is the same thing, by the position of the earth in its orbit; so that, to speak in general terms, the orbit of the moon becomes enlarged, in proportion as the earth is nearer to the sun, and contracted again when it is at a greater distance from it.

With this change of position of the system of the earth and the moon, therefore, all the quantities of these influences change anew, so as to form what is called the *Annual equations of the moon;* these changes evidently keep pace with what in the motions of the earth, or, apparently speaking, in the solar tables, is called the *Equation of the centre*, of which Table I. line 8, gives the maxima for all the planets.

§ 88. Under the name of the *Secular Variation* of the moon, is comprehended in, astronomy, the final results of the general influence of the sun upon the moon, in conjunction with the secular variations of the excentricity of the earth's orbit. This affects of course all the variations of any one of the elements of the orbit that have been mentioned before; it comprehends therefore, in some measure, the remaining total effect of them in a long period, reduced to their secular value, for more convenience.

§ 89. Though the more minute details and complications of influence resulting from the principles stated before, are of the greatest interest in practical astronomy; it may here suffice, to give only some general views, and make such remarks as may give the means of judging in relation to them with more ease.

It will be evident that all the variations and influences in the *Perturbations* of the *moon, or of its orbit*, will alternately act in augmentation or in diminution of the quantities to

which they are applied, or upon which they exercise an influence; but they will ultimately produce a successive retrogradation of the moon's orbit: that is, produce a motion of this orbit in the inverse order of the signs of the ecliptic. The changes of the inclination of this orbit will be oscillatory.

The attraction of the earth by the sun is more than double that of the earth upon the moon; if this were not so, the gravitation of the moon towards the earth, at the new moon, would be three times that which would otherwise take place between the moon and the earth alone; at the full moon, on the contrary, this action is to augment the distance of the moon from the earth.

The diminution of the moon's gravity in the syzigies being double of that in the quadratures, there is an excess of this effect which produces an advance of the line of the apsides.

In our times the moon's motion is accelerated; this arises from the action of the sun, under the secular variation of the excentricity of the earth's orbit; whilst the excentricity diminishes the acceleration will increase, and inversely.

Even the ellipsoidic figure of the earth produces changes, or variations, in the revolution of the moon, in consequence of the preponderating action of the mass at the equator above the sphere, whose radius is the polar axis; these being determined both by theory and by observation, give a means whence to deduce the ellipticity of the earth itself; for this they give $\frac{1}{304,5}$ and $\frac{1}{305,05}$ by two different equations, in near coincidence with the results of actual measurements on the earth, when these results are combined so as to give the least possible difference; and this determination presents the result of the general effect of the figure of the earth, without any influence of the local inequalities which may exist in the figure of the earth.

§ 90. Taking into general consideration the effects of all these combined influences it is clear: that they must depend

essentially upon three variable circumstances, or elements, and their combinations, namely:

1st. The distance of the sun from the earth.

2d. The distance of the moon from the earth.

3d. The position or inclination of the line of the attraction of the sun to the orbit of the moon.

All these are variable, and their variations, together with the reciprocal influences of those variations themselves, are combined in every possible manner, within certain limits; producing alternately the greatest or smallest effect in the place of the moon, and ultimately a joint effect upon the motion of the moon's orbit in general; and this itself is subject to oscillations.

Thus, for instance: the combination of the elements which will produce the greatest effect in extending the radius of the orbit, or the evection, will take place when in an inferior conjunction of the moon, that is, at *New moon;* the *node* of the moon's orbit, and the *apogee* of the moon's orbit, are also in the same place, and the moon has no latitude, or is also in her node with the ecliptic; for then, being in the most remote possible position from the earth towards the sun, and the action of the latter the most direct, it is the most exposed to its influence; and the effect is exercised without diminution in the direction of the produced line of the radius of the moon's orbit. So far as this effect is concerned, the smallest radius would take place under the combination of the following circumstances: the moon in opposition or full; her latitude 0°; and the perigee and the node in the same point; all these would, however, be modified by the position of the earth in its orbit, and the moon would be in the nearest possible position to the earth, when, in addition to the foregoing circumstances, the distance of the sun and earth is greatest, or the sun in apogee. From these extreme cases to the zero of this effect, which, as stated, takes place near the quadratures or quarters of the moon, under the complete combination of all circumstances, it is hardly necessary to say that the combi-

nations of the temporary effects, which determine any intermediate situation, are varied in a very great degree.

§ 91. Such would be the result of these different effects, if their causes were to produce them instantaneously; but the maxima or minima and the gradual increase or decrease of all these effects, as to their appearance, depend on a succession of continued actions of the attraction, which need time to reach their accumulated and full effect, and to become apparent to us; therefore, all the phenomena, as above stated, correspond, when observed, to places and times different, and retarded, from those stated, as their mathematical point of action; and for this retardation it again becomes necessary to account in the calculations of the places of the moon, for any given moment.

§ 92. We have seen in speaking of the planets, and the figures representing their orbits show: that to determine the place of a celestial body, it is necessary to refer this position to certain determined points in the circumferences of two great circles, or planes, perpendicular to each other, for which the plane of the ecliptic, and the plane perpendicular to the same, passing through the celestial body itself are selected. The effect of all these influences is therefore to be considered in the same manner, and reduced to that which they represent in these two planes, or what is called in longitude on the ecliptic, and in latitude perpendicular to the same. Under the varied inclinations which both the orbit of the moon, and the direction of the sun's attraction have to these planes, the reduction to them again occasions new complications; still, it is in this form that they are all recorded and treated of in the tables of the moon's motions.

§ 93. The proximity in which the moon presents to us all these varied effects of universal gravitation reciprocally crossing each other in their periods, and influencing each other in various ways, renders them greater and more obvious to observation than the corresponding effects in any other part of our solar system; they are, therefore, more difficult to deve-

lope from each other, to submit to regular calculation. At the same time they furnish the means to perfect the theory of this mutual influence of the celestial bodies on each other's motions; and from unaccountably irregular, as the motions of the moon appeared to the earlier astronomers, the present advanced state of the science, with the assistance of a highly cultivated mathematical analysis, have brought the results of the theory and the observations to a highly satisfactory coincidence, and derived from it great results, not only for theoretical science, but for the important solution of the problem of determining the position of places upon the earth in respect to longitude, the solution of which, was wanted in order to determine precisely the position of places on the globe, for which the latitude alone is insufficient.

From this same proximity and the apparent magnitude of the perturbations in the moon's motion, it necessarily arises that her apparent diameter is so sensibly affected as to require their influence to be taken into account; and this is still more necessary in the case of the lunar parallax; for this last is much greater, inasmuch as it is equal to the semi-diameter of the earth, as seen from the moon, which is nearly one degree. It is evident that this latter quantity must even be different for different radii of the earth, which decrease from the equator to the pole. We shall see this more clearly in its place, but we may be aware of it from the fact already stated that the earth is an ellipsoid and not a sphere.

§ 94. We have seen that the sun is the origin of all the perturbations of the moon above enumerated; and from this arises the necessity, when it is desired to determine the place of the moon, to determine previously the apparent place of the sun with accuracy, in order to obtain the elements of the changes of the moon, which will correspond to that situation.

§ 95. Since the theory of the moon's motion has been rendered more perfect, and the knowledge of her position has become in some measure a daily want, not only in science, but in the practice of navigation and geography, her posi-

tion, the temporary rates of her motion, her diameter parallax, &c. are marked in astronomical ephemerides, at intervals not greater than three hours distant from each other; not only these but even her direct distance from the sun and several well determined fixed stars, are given at the same intervals of time, for the immediate use of the navigator, &c.

§ 96. A remarkable difference between the planets and the satellites lies in this circumstance: that, while the former have all a rotation around an axis within themselves, independent, and, as far as we can yet see, not proportioned to the time of the revolution of the planet, we find on the contrary that the moon turns always the same side towards the earth. From this it arises that the duration of a day and night upon the moon, as dependant upon its rotation, and consequently also the position of the sun and that of the earth, for any particular point of the moon, is confounded with the synodic revolution of the moon; that is, they are of the same duration; and to speak the same language as we would apply to the earth, the year and the day are of the same duration in the moon.

Another result of this circumstance is, that the same hemisphere of the moon is always turned towards the earth; it is in fact well known that we always see the same configuration upon its surface, under equal quantities of illumination, but within the limits of certain small oscillations which will be mentioned hereafter. As a consequence of this, we find, from the principles of attraction, that the moon must be of an ellipsoidal, or rather oval form, elongated in the direction of the earth. And if, from the small density of the moon, we should allow ourselves to ascribe to it a capability of yielding to a variation of this attraction, we might conclude a variability of the shape of the moon, according to its position in relation to the sun and the earth, on account of the unequal effect of the attraction upon the two hemispheres; of this, however, we have, or perhaps can have, no indication from observation, and this phenomenon must remain invisible to us, because we

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always see the projection of the great circle perpendicular to this elongation.

For the same reason, also, the earth is only visible to one half of the moon; the effect of which must make a great difference between the two hemispheres of the moon, as the part turned from us must be forever deprived of the reflected light of the earth; and, therefore, of any relief of the darkness of its long night, by a phenomenon similar to our moonlight, but much superior in magnitude and probable effect; while the part turned towards the earth has a regular, successive, and proportional exchange, between the direct light of the sun and the reflected light of the earth.

§ 97. The moon being by far the nearest to us of all celestial bodies, increased optical means have enabled us to observe the surface turned towards us, with great minuteness; we have as yet, however, derived from this contemplation rather food for the imagination, than any actual and precise knowledge of facts, in respect to the details of its physical constitution.

We see upon the moon's surface what we call spots; that is, parts reflecting less light than others; and these differences of light seem to be constant in their form, and without perceptible variation in their light. The different parts, or figures, which they present to us, have received names, and their positions have been determined by latitude and longitude, like the places on earth, and perhaps more accurately than some of our cities. We have a *Selenography*, (so the geography of the moon is called,) as we have a geography, and maps made with great care and precision. The parts distinguishing themselves by peculiar light or shade, have received the names of mountains, volcanoes, seas, marshes, according to the ideas and appearances which they presented to the observers. There the astronomers have recorded the memory of those men of former times, whom jealousy and ignorance hardly allowed to mention among those who disturbed the human society, and retarded its advancement

towards civilization, by the broils of their ambition, or their sordid stupidity, in the history of the misfortunes of the human race—its boasted political history. On entering the moon we find ourselves in the society of the philosophers, whom the vulgar insult, while they enjoy the advantages obtained by their talents and intellectual exertions.

If any general consequences are allowed to be drawn from the observations of the moon's surface, in this respect, we may draw the general inference:—that far from finding any reason to refuse to it inhabitants with intellectual, and, we will hope, *moral* faculties, all observations equally lead to the supposition, that physical circumstances and means, not much different from those on earth, must have aims and results not far different from those we here observe; and nothing can prompt the ignorant to refuse this extent of power and benignity of purpose to the deity, except the fear of seeing their arrogance humbled before the qualifications of beings to whom they might not dare to compare themselves.

§ 98. By the attentive observation of the effect of the twilight upon the part of the moon bordering the illuminated part, it has been found to extend over $2° 34' 25''$ of the circumference of the moon; this gives to its atmosphere an elevation of 0,0002533, or $\frac{1}{39446}$ parts of the radius of the moon; from this observation, we evidently obtain only such an arc, and consequent elevation of the atmosphere, as will intercept a sufficient part of the solar light to produce an obscurity discernible by us.

The small density of the moon must also naturally lead to a lesser density of the atmosphere than ours, as has already been observed, in this respect, for the sun; in addition to this, we must also expect a smaller extension of this atmosphere, in consequence of the small mass of the moon.

The refraction of light in the passage through aëriform or gazeous substances, being only a fractional effect of their density, we must not expect that the observation of this effect can ever serve as a very accurate mean of determining the

atmospheres of celestial bodies, either in extent or density; and it would be of little interest to make here a supposititious calculation upon this subject. However, the observations hitherto made lead to a horizontal refraction of $3\frac{1}{2}$ seconds, which, considering the small difference of density between the sun and moon, appears to come in support of the ideas given above upon this subject.

§ 98. We are already habituated to see every movement of the celestial bodies affected by periodic changes, or oscillations on both sides of a certain mean state; whether we refer to the actual motions of heavenly bodies, or to the appearances which they present to us; such is also the case with the appearances of the moon's disk, which, by its proximity to the earth, presents always less than 180° of its circumference to an observer on the earth's surface; for instance, under a diameter of 32′, only 179° 28′ of its circumference. With a nearly apparent equality of the general appearance, this disk is however subject to changes, observable only on the borders, and depending on various causes; these changes are called *Librations*, and are as follows:

1st. *The libration in longitude.* We have stated, that the revolution of the moon is equal to what might be considered a rotation around its axis, in relation to the sun; because it occasions every part of the moon to be successively presented to the sun. But, as we have seen to occur upon the earth, in respect to mean and true solar time, the different velocity with which the moon arrives successively in the successive superior conjunctions, or full moons, where we see it in full light, occasions that these full lights present us with a surface of the moon, varied in proportion to this difference of coincidence between the true and mean return to the full moon; this change progresses therefore only a small step, as we might say, in each revolution of the moon, or what we call month; it depends on the revolution of the moon's orbit itself, or what is called the line of its apsides; the revolution of these we have stated to take place, in relation to the sun,

in a period of about $3231\frac{1}{3}$ days, or about 118 changes of the moon; the magnitude of this change of appearance amounts to about 8° of the moon's equator, and is sensible only in a small change in the appearance of the spots in the moon, east and west, near the equator, if we may thus express ourselves. The whole change of appearance, therefore, returns in the same order after the period stated; and since the time that observations have been made upon the moon, accurate enough to determine it, there has not been found reason to suppose it productive of an ultimate progressive change.

2d. *The libration in latitude* is the apparent effect which the inclination of the orbit of the moon to the ecliptic produces to a spectator in this plane; the moon being alternately north or south of the ecliptic, in the course of its revolution, we naturally see more of the *southern* or *northern* hemisphere in proportion to this deviation of the moon in latitude *north* or *south;* the epoch of this libration must be that of the revolution of the moon in relation to its nodes, and its magnitude must of course, on each side of the mean situation, be equal to the inclination of the lunar orbit to the ecliptic, or about 5° 8′.

3d. *The third libration* is similar in its effect to the nutation of the axis of the earth, as we have described it, and of course affects both the preceding apparent nutations. The inclination of the equator of the moon to its orbit, and of this to the ecliptic, furnish three planes, intersecting each other in the centre of the moon, which are found to have one common mean node; of these three planes, that parallel to the ecliptic is always between the two others; it makes, with the lunar equator, an angle of 1° 30′ 11″, and with that of the lunar orbit, the angle of 5° 8′ 38″, as given before; their common node is affected by a retrograde motion, the period of which is 6793,391 days; during this period, the pole of the lunar equator and that of its orbit, describe a circle around the pole of the plane parallel to the ecliptic, in such

a manner, as always to preserve their respective position in the same great circle of the sphere. The period of this nutation is subject to various inequalities, and follows, besides, the variations of the inclination of the moon's orbit.

4th. *The daily libration* is the effect produced by the change of the parallax of the moon. On its rising, the observer stands upon the end of the earth's radius, perpendicular to the line of centres; he sees therefore, on the surface of the moon, farther than the radius perpendicular to this same line, by the amount of the difference between the moon's parallax and its radius; both these quantities change in magnitude, and the position of the plane, in which the parallax changes likewise, becoming smaller as the moon is more elevated, and increasing again after the greatest altitude in another direction, other parts of the moon become visible in consequence of this, which forms the *daily libration*, and is evidently only sensible on the edges parallel to the horizon.

It is evident from all that has been said in relation to the motions of the moon, either real or apparent, that all the epochs and magnitudes of these librations are subject to oscillatory changes, or differences, which, in fact, do away with all epochs, and reduce their determinations, like all others in the present more advanced state of astronomy, to the mere calculation of the real or apparent state from the data, grounded only upon the general epochs of the mean motions of the heavenly bodies concerned.

§ 99. The variation of the appearance of the illuminated part of the moon forms such a striking phenomenon, that no spectator can have failed to follow it in all its details; it therefore hardly needs to be minutely explained, as the successions of this illumination in every month follow so evidently the position of the moon in respect to the sun.

The figure in the middle of PLATE III. gives an idea of the successive state of illumination, which the moon must present to the earth in the course of its revolution, according to the angle under which we see it from the earth; the illuminated

part of the moon is always that which lies within the circumference of the small sector of a circle, drawn around the sun with a radius equal to the distance of the moon from it, while we see from the earth always that part of the moon which lies within the circle around the observer on earth, drawn with the radius of the moon's orbit, and which includes so much a greater part of the enlightened hemisphere of the moon, as our line of vision coincides more with the cone, or line, of illumination from the sun. Opposite to each place in the moon's orbit is marked, in the figure, the appearance which that situation presents to us.

When the moon is in the straight line between the sun and the earth, or very near it, the dark part of it being turned towards us, the great light of the sun renders the faint illumination, which the reflected light of the earth must occasion in the dark part of the moon, imperceptible, notwithstanding this illumination is then the greatest; then of course we do not see the moon; but the motion in its orbit in one day bringing it to a distance of about 13° from the sun, it becomes visible, by shewing the illuminated part of an equal number of degrees on the side towards the sun; the fainter light occasioned upon the dark part, by the reflection from the earth, renders the whole surface of the moon discernible, in a favorable state of the atmosphere.

This beginning of the appearance of the moon, which is called the *New moon*, occurs, as in the case of all the planets, when she first becomes visible, after her proximity to the sun, in the western horizon, after sunset. The time of the stay of the moon above the horizon after sunset, and the magnitude of the illuminated part of the moon, alike increase rapidly every day; the first about three-quarters of an hour every day; the latter about 13° of the moon's circumference; so that in about one week the moon reaches a position about at right angles to the sun, and an illumination of about half its apparent disk. With the augmentation of the angular distance of the moon from the sun, the illuminated part of the

moon becomes more visible, and the fainter illumination of the dark part, by the reflection of the earth, becomes gradually entirely obliterated. The moon, having arrived at the position in its orbit directly opposite to the sun, and therefore rising at about the same time when the sun sets, it appears completely illuminated, and embellishes the spectacle of the heavens the whole night: it is then *Full moon*. From this position, the moon, continuing her apparent course in the same direction, begins to remain above the horizon in the morning after the rising of the sun; and at the same time approaching to the sun on the opposite, or eastern side, gradually presents us daily more of its dark part on the west side, and resumes the appearances which it had presented between new moon and full moon in an inverted order, during its revolution between full moon and the new moon, which brings it again to the proximity of the sun, whence we began the description of its appearances.

CHAPTER IV.

Of the Eclipses of the Sun and Moon.

§ 100. In consequence of the revolution of the moon's orbit, and its inclination to the ecliptic, new and full moon will occur successively in different parts of the moon's orbit; and, therefore, at different distances from the plane of the ecliptic.

When the *New moon* takes place near the nodes of the lunar orbit, the apparent angular distance of the centres of the sun and moon may be less than the sum of their semi-diameters; and if such is the case, as seen from the centre of the earth, or even from any point on the earth's surface, it is evident, that the interposition of the moon between us and the sun will make it appear as projected upon the sun; this produces what is called a *Solar eclipse.*

The earth, receiving its light from the sun, must necessarily project on the opposite side a cone of shadow: this extends farther than the orbit of the moon, as may be easily shown by calculation. If, therefore, at the time of *Full moon,* the distance of the centre of the moon from the axis of this cone of shadow is less than the sum of the radii of the moon, and of this cone of shadow at the place where the moon passes it, the moon will be darkened by this shadow. This passage of the moon through the earth's shadow constitutes a *Lunar eclipse.*

As the light of the sun does not emanate from one single point, but from all the points of an extensive surface, each of which would form its own cone of shadow behind the earth, the crossing of all these cones will form, not only a cone

whose base is turned towards the sun, which terminates at a certain point behind the earth, and whence the whole of the sun's light is excluded; but also a truncated cone whose section is constantly increasing behind the earth, and which includes every gradation of shade according to the greater or less portion of the sun that is apparent from any point at a certain distance from the axis of the cone. This is called the penumbra, and its limit must be very indistinct and uncertain. It may evidently happen, that the moon passes only through this cone of the penumbra, and may only suffer a partial loss of its light.

§ 101. *A lunar eclipse* being an actual event, or phenomenon, not a mere appearance depending on the relative situation of the spectator, happening therefore at an absolute moment of time, at which different observers in different places upon the earth, will be in different points of their rotation in relation to the sun; these different observers counting, as we have seen to be a general principle, each his time from his position in respect to the sun, they will, as we may say, register the moment of this event at different times; and this difference of the time, counted at two places at the same absolute moment, constitutes their difference of longitude. Thus, lunar eclipses would be the easiest mode of determining the difference of longitude upon earth, if their observation were capable of that degree of accuracy which the present state of the science requires; they were, therefore, in the infancy of astronomy, before better means, and more ability in the necessary calculations, were at the disposal of astronomers, as valuable as they are now scientifically uninteresting, and, in consequence, neglected. That the moments of contact of the moon with the actual cone of the shadow of the earth must not be very distinct, will easily be concluded from the natural circumstance, that from the outside of the penumbra to this cone the light must diminish by a gradation corresponding to the magnitude of the sun's disk shut out from the moon, upon which the apparent darkness depends.

The radius of the section of the cone of shadow, at the place of the moon's passage through it, is always equal to the difference between the sum of the parallaxes of the sun and moon, and the radius of the moon; and these depend on the distance of the moon from the earth, and the distance of the earth from the sun. These quantities determining the latitude of the moon at which an eclipse can take place, correspond to a certain distance from the nodes on either side, and the limits of their variation determine the limits of the distance from the nodes in longitude, at the time of the conjunction of the moon, in which an eclipse of the moon is either certain or possible; at a greater distance the eclipse is impossible.

The certainty of an eclipse of the moon is within 7° 47′ of distance from the nodes; and the impossibility commences at 13° 21′, and excludes all greater distances.

The duration and magnitude of these eclipses depend on the same elements; and the first of these, besides, on the velocity of the moon at the moment, by which it would of course pass the section of the shadow at a quicker or slower rate. The diameter of the cone of shadow at the place of passage of the moon varies between 37¾ minutes and 46⅓ minutes. All these elements must, therefore, be taken into account, if an accurate calculation of an eclipse is required, for the purpose of predicting it, with its peculiarities.

The cone of the actual shadow of the earth is, at a mean, about 110 diameters of the earth in length, or 36 times the radius of the moon's orbit; the vertical angle of this cone is 31′ 13″. These elements of course vary with the distance of the earth from the sun. The moon's distance varies with the very variable ellipticity of its orbit. Certain combinations of the apogee both of the sun and moon, and the influence of the refraction of the atmosphere of the earth may occasion the moon to pass only through the penumbra, and the shadow will be modified, by the passage of the light, from the part of the sun not actually covered for the moon, through the atmosphere of the earth. The shadow, or apparent colour of

the moon, in this case, having been observed to have a redish hue, we may conclude that the earth's atmosphere admits the passage of the red rays of light in preference, as the light thus mixed with the shadow must pass in a great part through our atmosphere. If the eclipse should take place in the greatest proximity of the moon to the earth, and of this to the sun, the moon may be plunged so entirely in the full shadow of the earth, as even to become invisible.

The figures 6 and 7 in PLATE V. which are intended for the explanation of the eclipses of Jupiter's satellites, also give an idea of lunar eclipses, the phenomena being exactly similar, with this difference only: that the smaller diameter of the earth, and its greater proximity to the sun, occasion the cone of the true shadow of the earth to be so much shorter, and the penumbra, on the contrary, to spread so much wider. Besides, by our proximity to that phenomenon, we can observe details which become obliterated at the greater distance at which we see the occultations of Jupiter's satellites.

§ 102. The passage of the moon between the sun and the earth, sufficiently near to the line joining their centres, *to intercept from any part of the Earth the light of any part of the Sun* produces *a Solar eclipse*; this furnishes one of the most interesting and most important observations of our astronomy.

This phenomenon is therefore properly rather an optical projection of the moon, or part of the same, upon the apparent disk of the sun; such as will refer to the visual lines drawn from the point of observation on the earth's surface, to the respective parts of the moon's circumference, and prolonged to the apparent disk of the sun. From this point of view it will be most proper to deduce the general considerations which we have to present upon this subject. We shall only state as facts the numerical data which may be needed, the details of these belong to calculations into which it is not our object to enter.

Every point of the earth's surface presents itself as the vertex of a cone based upon the great circle of the moon perpendicular to the axis of vision passing through the centre, and the investigation to be made is, whether and where the prolongation of this cone beyond the moon will meet the sun.

The moment for which this fact is principally to be ascertained is that in which the moon passes the circle of latitude, drawn through the centre of the sun, which is called conjunction, because near that place and time the greatest proximity of sun and moon will take place.

The general phenomenon is of course here again, as in all cases, referred to the centre of the earth, and the calculation of this appearance, for any point upon its surface, must evidently take into view the place this point occupies in its daily rotation in relation to the sun, and its position upon the earth's ellipsoid, upon which the parallaxes depend. These elements are constantly changing; and their effects are so much more complete as the changes take place in directions crossing each other variously, as well as different from that of the relative apparent motion of sun and moon. The diameters of the sun and moon are equally variable, especially the latter; they form the quantities which are to be compared with the apparent positions of their centres. These variations of all the elements of the calculation oblige us to ascertain their corresponding values for every moment of the duration of the phenomenon, the length of which is of course dependent on the proportion between the magnitude of the part of the sun covered by the moon in its passage, and the relative excess of the moon's motion over that of the sun; it is therefore usual to calculate these elements and their variations for every hour in the neighbourhood of the time of the phenomenon, whenever it is to be foretold for a certain place, or when the result of actual observations are to be applied.

§ 103. In Plate V. figure 4, the section of the cone drawn from the centre of the earth tangent to the moon, and

of which the dotted line is the central axis, being supposed to meet the sun, represented by the greatest circle, will project the moon to an observer in the centre of the earth, at the moment of conjunction, in the circle: L, I, K, H.

It must be observed, that, for the sake of ease in representing the appearance we have taken the liberty to represent these projections in the figure as circles, although, as they are perpendicular to the plane upon which the figure is in other respects drawn, they would become mere straight lines.

This circle, L, I, K, H, would lie upon the sun, except the part L, I, K, A. An observer at the point where the axis of the cone (the dotted line) cuts the circumference of the earth, would see the eclipse at the same place without any influence of parallax, and with the sole difference of an increase in the diameter of the moon proportioned to the diminution of the distance of the whole radius of the earth; he would see the phenomenon exactly in his zenith; such a point can therefore only be within the tropics, and furnish a very limited case.

An observer in the point marked t, in the circumference of the circle representing the earth, will of course see the moon projected in the circle, C A. &c. completely within the sun, which produces what is called an *Annular eclipse*, and it is easy to see, that, if by the changeable ratio between the diameters of the sun and moon, the latter happened to be larger, he might see no part of the sun, and therefore have what is called a *Total eclipse*, even of some duration. The phenomenon is of course affected by the parallaxes of the sun and moon, and the augmentation of the diameter of the moon, due to the difference between the centre of the earth and the point t, in the position in which it may then be in its daily rotation, and in respect to the sun and moon.

If we remove our observer to the point t, the moon will appear to him in relation to the sun, represented by the circle D, F, L, E, K, which will hide only the part I, L, E, K, of the sun, and the same eclipse which appeared annular to the

observer in t, will be only small for him; and this difference will be due to the apparent effect of the parallax, and the change of the apparent diameters due to the point t, compared to the point t. Such will evidently be the case for any other point of the surface, every observer seeing the moon, projected differently, also sees the eclipse differently, or what is usually called *different phases* of it.

What has been said here for the moment of conjunction, evidently applies to every other moment of the phenomenon, and therefore to the beginning and to the end. The eclipse will begin for any point upon the earth at the moment when the distance of the centres of the sun and moon will be equal to the sum of their semi-diameters; and it is easy to see that this line of the centres or semi-diameters will be differently inclined to the directions of the apparent motion of the sun and moon, according to the place of observation, and the direction in which the moon passes, or the apparent angle between the orbits of the sun and moon; the calculation of the effect of the parallax for any place and time is, therefore, the principal subject to be attended to.

In PLATE V. figure 5, is represented the appearance of a transit of the moon before the sun, as it would appear to an observer in t; and it is easy to conceive, that the change of position of the observer, and that of the apparent position of the moon's orbit, will by a combined effect vary the line of the apparent path of the moon in front of the sun: the first by the effect of the parallax of the moon, the second by the variation of the position in which the eclipse takes place. The moon must evidently approach the disk of the sun from the west, and by the effect of its greater angular velocity, it will pass to the east side of it in such a time as will correspond to the combined effect of the excess of its velocity over that of the sun, and the cord of the sun which it has to pass, affected by the parallax corresponding to the place of observation.

§ 104. The great distance of the sun in comparison to that

of the moon from the earth, occasions that the angle subtended by the sum of the radii of the earth and moon, together with its penumbra, is always less than $\frac{1}{4}$ minute, as it is equal to the sum of the parallaxes of the two bodies; this angle is the deviation from the parallelism of the two lines drawn from the sun to the earth, or to the moon; and, therefore, its influence upon the appearance of the eclipse is only small, and the limit within which a solar eclipse must, or may, take place, is, generally speaking, determined by the sum of the apparent diameter of the moon and its parallax, which may at a mean be stated at 90′, or $1\frac{1}{2}$ degrees. With this, therefore, the latitude of the moon in any new moon must be compared, to determine whether an eclipse can take place, and the difference of longitude of the sun and moon, corresponding to such a latitude, will, therefore, give what is usually called the mean limit of such an eclipse. The variation of these two quantities furnish the following more detailed results:—The *certainty* that a solar eclipse *must* take place at some point on the surface of the earth, is limited by a distance of 13° 33′ from the moon's nodes with the ecliptic: the eclipse ceases to be possible at a distance of 19° 44′ from the nodes; between these two limits, a more close investigation of the circumstances and elements must determine whether an eclipse will occur or not. These limits including the whole surface of the earth, are of course not those that fit any particular place; for that individual case they become of course much narrower, as they not only become reduced to one point, instead of the whole semi-surface of the earth, but the appearances are combined in a variety of ways, according to the position of the place and the course of its rotation with the earth, compared to that of the moon in relation to that of the sun. From this circumstance it arises, that though the actual phenomena of solar eclipses, generally speaking, are more numerous than lunar eclipses, these latter are, notwithstanding, more frequent at any given place of the earth. It is evident, that the influence of the presence or absence of the phenomenon above the

horizon of the place has its influence in both cases; and even the refraction. By this it may happen that an eclipse of the moon may be visible even during the presence of the sun over (but near) the horizon of a place. When a solar eclipse appears annular at any place, it is evident that this indicates that the cone of the shadow of the moon does not reach that place: if, on the contrary, the eclipse appears total, the place is actually within the cone of true shadow; in the other, or partial eclipses of the sun, the place where the observation is made lies actually in the penumbra of the moon. These circumstances show, that the cone of shadow has an approximate length, at a mean very near equal to, and varying more and less with, the distance of the moon from the earth. By these various combinations, the longest possible duration of a total eclipse has been found to be $7^{m.}$ $58^{s.}$, and the longest duration of an annular eclipse of $12^{m.}$ $24^{s.}$. It is easy to see, that a case may occur such as to occasion an eclipse of the sun, seen at a certain place upon the surface of the earth, to be no more than a conjunction, as seen from the centre, and without an eclipse.

By the effect of the motion of the moon's orbit, the eclipses of the sun and moon will return very nearly in the same order, in a period of 223 synodic revolutions of the moon; because 19 revolutions of the node coincide very near with this number of revolutions, as 6585, 32, and 6585 days 78, leaving only 0,46 of a day difference, consisting of 18 years 10 or 11 days, which period gives about 70 eclipses, namely, 29 lunar, and 41 solar eclipses.

§ 105. In any use to be made of the observations of a solar eclipse, it is always necessary to reduce the result to the time, and to the other corresponding circumstances which it would present for the moment of conjunction, at the centre of the earth. Thus reduced, the observations made in different places become comparable, and furnish determined points, and fixed epochs, for the apparent relative motions of the sun and moon, from which their actual revolutions are calculated, and

thus the tables of their motion and revolutions may be constantly improved. This conjunction happens at a certain absolute moment of time, which, as we observed already of the lunar eclipse, is registered, we may say, by every observer, in terms of the time shown at his own place, and this time is regulated, as we have seen, by the point which the place of observation occupies in the course of the daily rotation of the earth; we are therefore enabled to determine in return, from this absolute moment, and the difference of time shewn at any two places of observation, their relative position in the rotation of the earth; that is, their *Difference of longitude*, because the present perfection of astronomical tables admits these comparisons to be made with the greatest nicety. Since this perfection has been attained, solar eclipses are the most accurate means of determining the longitude of places on earth. The distinctness of the limbs of the sun and moon, the velocity of the motion, the general ease with which these observations can be made, concur in assisting the observation, and increasing this accuracy.

The great achievement of astronomy, the calculation and prediction of a solar eclipse, is one of the first signal services rendered to humanity by this science, by liberating man from the terror into which he was before thrown, on seeing his Deity suddenly and unexpectedly taken away from him, and vanquished, as was imagined, by the monsters, his enemies. What state was left for man to expect if his god was conquered? How different are now the feelings of civilized man on anxiously awaiting the predicted second of time in which he expects the enjoyment of seeing his calculation verified, and his exertions rewarded, by the assent of the heavenly bodies to theories of his own creation?

§ 106. We may be allowed here to step beyond the limits of what has hitherto been explained, to give a general idea of a phenomenon similar to the solar eclipse, called an *Occultation of a fixed star* by the moon. We may suppose our reader acquainted with the existence of the fixed stars, as

points apparently steady, much more distant than the moon; and the first contemplation of their number might even cause us to wonder that the event of the moon intercepting the view of the one or the other of these stars is not more frequent than it actually is. But among the many chances which the varied and winding course of the moon must occasion, it is evident that the phenomenon must occur, though the apparent diameter of the fixed star reduces the distance from the central position of the moon, within which the phenomenon can take place, to the radius of the moon alone, and the effect of the parallax of the moon, when the whole disk of the earth presented to the phenomenon is considered. The phenomenon of the *Occultation of a planet* by the moon, is evidently among these chances, and the smallness of their apparent diameters render it very little different from the occultation of a fixed star.

These phenomena are of course not limited to a certain position of the moon relative to its node; in all other respects they are similar to solar eclipses in their calculation, if we were to suppose, in what has been said above, the sun placed freely in the path of the moon, without regard to the ecliptic, and without allowing it any diameter. The conjunction generally taking place at a greater distance from the plane of the ecliptic, alters only the proportional reduction to that plane, and the central conjunction in longitude.

These phenomena therefore furnish fixed points, and epochs of reference, for the moon's motion in relation to the fixed stars; they may equally well be transferred to the moon's motion relative to the sun, as the results of the solar eclipses upon its relative motion, may be to that relating to the fixed stars. That they equally multiply the means of determining the longitude of places on earth, is clear of itself, for the appearances being in this case again reduced from any place of observation on the earth's surface, to the line of the centres of the earth and moon, and its projection on th ecliptic, or what is the same thing, to its correspondir

latitude and longitude, the time corresponding to the conjunction of the moon with the circle of latitude passing through the star, gives an absolute moment of time, to which the special times of different points of observation correspond; and the difference between these is, as we know, their difference of longitude on earth. In respect to the perfection of our knowledge of the moon's motions, these phenomena furnish of course very valuable results relative to the inclination and the various motions of the orbit, under variations of circumstances and positions which have peculiar value.

CHAPTER V.

Of the Planets exterior to the Earth in the Solar System.

§ 107. On leaving the earth and its satellite, the moon, to proceed to a greater distance from the sun, we meet, at about one and a half times the distance of the earth from the sun, the planet *Mars*, which performs its revolution around the sun solitary, unaccompanied by any satellite. His *apparent diameter* must evidently be very variable, as we see him in his opposition to the sun at only one fifth the distance he was when near his superior conjunction; in this last position, however, he is invisible to our present optical powers. This diameter, when Mars is in opposition, is 18″,3 and it diminishes to about 3″. The mean diameter is about 9″.

His *real diameter* is little more than half that of the earth, and his other dimensions proportional; the mass is only about the seventh, or eighth, part of that of the earth, while the density resulting from this is, from the first somewhat more, and from the latter somewhat less than that of the earth. The first of these dimensions are the results of the latest calculations upon the perturbation occasioned in the revolutions of the earth by this planet, for the absence of a satellite deprives us, as in the case of Venus, of the means of direct determination.

Mars distinguishes himself from the other planets by a less brilliant light, of a reddish hue. According to the theory elucidated in speaking of the sun, we might consider this as

a result of his less mass and greater density, which would endow him also with less proper heat and light; at the same time his distance from the sun deprives him of the advantage of the denser light of the sun, which Venus and particularly Mercury, notwithstanding their small mass can reflect to us.

§ 108. The day of Mars, or his rotation upon his axis, is nearly 40″ longer than a day of the earth, and the diameter being smaller the velocity of a point in the equator is much less than upon the earth, his ellipticity is assumed at $\frac{2}{191}$.

The inclination of the plane of his equator to that of his orbit being 28° 38′, while that of the earth is less than $23\frac{1}{2}°$ the variation of the seasons upon Mars must be greater than upon the earth.

Mars performing his revolution outside of the orbit of the earth, the visual ray from the latter can never make a tangent to his orbit, therefore he can never present to us the variety of phases which the inferior planets do; still we observe some change in the appearances, as must necessarily follow from our proximity to him, and the different positions in which we see him in relation to the sun. The dark part of Mars can never exceed the angle under which this planet will see the earth in her greatest elongation, that is between 41 and 42 degrees of the planet's circumference. This circumstance affects the apparent diameter of the planet; which, besides, presents itself under various inclinations, arising from the inclination of the orbits of the Earth and Mars; and with this, the varied inclination of these towards the equator, in parallels on which all the heavenly bodies pass daily before us, concurs to render the determination of his diameter difficult and complicated, except in his opposition to the sun, when we see him, of course, in full light.

Observations with great magnifying powers, have discovered inequalities upon the surface of Mars, of considerable apparent magnitude, but generally speaking we are in considerable ignorance in relation to the more minute details of his constitution.

§ 109. The four small planets of modern discovery, which are next met with after Mars in our solar system, *Vesto, Juno, Ceres* and *Pallas,* have presented us with much novelty in their revolutions, arising from their great eccentricity, the great inclination of their orbits, and the particular manner in which they intersect each other, subjects of which we have spoken in the proper place. They exhibit to us apparent diameters so small, that our powers of vision, although so far increased beyond what was considered possible a century since, as to furnish the means of these discoveries, have not yet been able to inform us of any peculiarities they may possess.

For the same reason they appear by no means brilliant, but, like fixed stars of from 5th to 9th magnitude; it has not yet been possible to observe their *rotations,* still less any other peculiarities. The greatest apparent diameter of Juno is little more than 3″, that of Ceres, 2″,5, but that of Pallas may amount to 6″,5, and Vesta, that appears the smallest, has not yet been determined. These determinations, however, indicate for Pallas, a real magnitude greater than that of Mars, being near two-thirds of that of the earth. Ceres presents, sometimes, a nebulosity which exceeds her diameter above 6″. This is a phenomenon otherwise peculiar to comets alone. The proximity of the orbits of these small planets to Jupiter, the largest planet of our system, occasion an influence of the latter upon their motions; and this is so much the greater, as the excess of his mass is greater, and so much more complicated as the respective positions of the orbits subject these to a greater variation of their distances. Thus, for instance, in the case of Pallas, four hundred corrections, or equations, occur in calculating a position, among which is one depending on Jupiter, that amounts to one whole degree. Thence the discovery of these planets has given occasion for a considerable perfection in the formation of the astronomical theories.

On the other hand, the perturbations of the course of Ju-

piter in his orbit, appeared so fully accounted for by the influences of Saturn, Mars, and Earth, that no suspicion, arising of any other influence, neither there was any suspicion of their existence then, nor has there been any influence upon the neighbouring orbits discovered since their discovery, though in principle this must exist, but be as yet below our power of accuracy in observation.

But it may easily be conceived, that to observe particulars upon bodies presenting themselves under such small angles is attended with no small difficulties, as great magnifying powers of telescopes are required, which equally magnify the apparent motion, so much that it becomes very difficult to follow the planet in its course, with sufficient steadines to allow close and sharp observation; the discovery of these details therefore appears to be reserved for the future improvements of astronomy. We have, therefore, as yet no determination of masses, densities, or other such data, the discovery of which we can only hope from the close investigation of their mutual influence in their respective revolutions.

§ 110. *Jupiter* the next planet in our system, after the four small ones lately discovered, is the largest of them all; and being accompanied by four satellites, or moons, presents us a whole system in itself. His greatest apparent diameter as seen from the earth, is 44″,5, the smallest 30″, while if we could see him at the same distance as the sun, he would appear under an angle of 3′ 06″,8. The sun, as seen from Jupiter, subtends a mean angle of no more than 6′. 3″,5. From the sun he must appear under an angle of 36″,3, or more than double the angle subtended by the earth, notwithstanding the distance of Jupiter is more than five times as great. The volume of Jupiter is near 1281 times that of the earth, while the mass is only 309 times that of the earth; this might again be considered as a strong proof, that in the celestial bodies, the heat is in proportion to the masses, and occasions an expansion of their mass, or diminution of density. This last is in Jupiter only between one-quarter and one-third of

of that of the earth, and little more than that of the sun.—He is also the most brilliant of all the planets; which appears to corroborate the general principles expressed on this subject in treating of the sun, for we may suppose, that this brilliancy, at so great a distance from the sun, where Jupiter receives only so small a cone of light from the sun as 36″,3, may in part be due to the state of the planet itself. Perhaps, in order to weigh and compare all these effects with more accuracy, we should take into consideration the whole of the system of Jupiter, with his satellites, rather than the planet alone, for we are already shewn, by the example of the moon and the earth, which we have discussed at length in the preceding chapter, that in all cases where any external influence upon such a system comes into consideration, the action is upon the whole mass of it, and in all mechanical effects, is referred to the centre of gravity of the system; and, reciprocally, the reaction is that of the system, as well in quantity as in relation to the point from which this action proceeds.

§ 111. Jupiter has, of all the planets, the most rapid rotation upon his axis, the day being only $9^{h.}\ 55^{m.}\ 34^{s.}$; he revolves therefore near two and a half times while the earth revolves once; hence the velocity of a point in his equator becomes twenty-six times greater than that of one on the equator of the earth; and it occasions a flattening of the poles of $\frac{10}{177}$, or about seventeen times as great as that of the earth; this velocity also occasions the fall of heavy bodies at his equator to be diminished by a one-ninth part of that at the poles.

The plane of the equator of Jupiter is inclined to that of his orbit under an angle of 3° 5′ 31″; the inequality of the seasons must therefore be very small, and each parallel of Jupiter must have almost a uniform season; while the number of his satellites must vary and repeat the agreeable phenomena of the moon on earth, in manifold combinations, maintaining a constant illumination similar to our moonlight, and repeating the phenomena of eclipses with considerable variety,

both in relation to the sun, and in relation to these moons among each other. Similar to what we have observed in relation to the equator of the sun and the planetary orbits, the orbits of the satellites of Jupiter are about equally distributed on both sides of the orbit of the planet.

The face itself of Jupiter presents most generally two, three, or even four belts of shade, nearly parallel to his equator; but as they are not constant, and appear often to have a movement of their own, they are not considered as belonging to the body of the planet itself, but rather as an atmospheric phenomenon upon its surface, to which the rapid rotation of the planet may have a tendency to give this peculiar form in the appearance of belts as we see it: other speculations have concluded, from their variability, an unsettled state of the mass of Jupiter, and that it is still subject to some alteration by the rapidity of his rotation. The same circumstances, of the small density of Jupiter, and the necessarily still smaller density of his atmosphere, will not afford us any opportunity to observe the refraction of his atmosphere, and we have as yet no traces that may direct us in this inquiry.

§ 112. Since the earliest discovery of the telescope, we have become the spectators of the direct system of celestial bodies which Jupiter presents to us, in imitation of the system of the planets, and have witnessed the details of the phenomena that must accompany it, viewed at a distance. Its movements take place in such a small space, and in periods so short, that we are enabled to study the general principles the more easily, as its proportional distance is peculiarly favourable, at least during a considerable part of the revolution of Jupiter or the earth.

The orbit of Jupiter being but little inclined to that of the earth, and as the orbits of these satellites do not deviate much from that of Jupiter, they present themselves to our view under the form of very elongated ellipses, approaching more nearly to a straight line, as the earth is nearer one of its nodes with the plane of Jupiter's equator.

The principal facts in regard to the satellites of Jupiter, as well those relating to the elements of their orbits as to the satellites themselves, are collected under their proper heads in TABLE III., as far as their small appearance at our distance from them has hitherto enabled us to ascertain. The difficulty, which it is easily seen a minute inquiry into these details must have, and the practical use which was immediately made, after the discovery of these satellites, in the determination of longitude on earth, by the observations of their eclipses in the shadow of their primary, corresponding to our lunar eclipses, have given to this inquiry a peculiar turn, and an interest which has lost much since the discovery of more accurate means for the determination of the longitudes, but which the progress of the science, and more perfect optical means, will review upon a more scientific basis.

The satellites, in their revolutions, follow exactly the same laws and consequences of attraction as the primary planets; therefore, what we have stated section 9, under the name of the Laws of Kepler, applies to them in full; hence these orbits are elliptic; but the situation in which we can observe them is not very favorable for the observation of this ellipticity, and therefore it has as yet been observed only in the orbits of the two outer ones.

§ 113. In PLATE III. the left side figure represents the semi-circumferences of the system of Jupiter's satellites in its proper proportional magnitude, compared with the earth and the moon's orbit, also with the magnitude of the sun; they are purposely thus united, in order to shew their several proportional magnitudes. The small innermost circle representing the earth, and the next Jupiter, the first satellite is very nearly at the same distance from Jupiter as the moon from the earth. The semi-circumference of the sun falls outside of the second satellite, and only the orbits of the third and fourth are at a greater distance from Jupiter than the radius of the sun; the fourth is distant about 1¼ diameters of the sun. Below

this figure will also be found a section perpendicular to the plane of the orbit, shewing the inclinations of the orbits and that of the axis of Jupiter, in a similar manner as PLATE II. represents in the case of the primary planets of our solar system in general. Above the figure is placed the appearance of this system as it appears from the earth, when viewed under about the greatest inclination.

The expression of the distance of the farthest satellite in the same unit as the distances of the planets from the sun, that is in the radius of the earth's orbit, which is found in the fourth column of TABLE III., is 0,0115, or hardly more than $\frac{1}{100}$ of that distance. This shews that, for instance in our PLATE I., where the radius of the earth's orbit is half an inch, this whole system would be represented on the same scale by five thousandth parts of an inch, or by a mere point.

These satellites revolve with considerable rapidity in orbits of small inclination to the equator of Jupiter, which all lie between this equator and the orbit of Jupiter.

As seen in TABLE III., the first satellite performs one revolution around Jupiter in about one and three-quarter days, the second in about three and a half, the third in seven days and about four hours, and the fourth in about sixteen and three-quarter days; the inclination of the orbits to the equator, as equated, are of the First, - - - - - 6″,5

Second, - - - - -	1′ 5″
Third, - - - - -	5′ 1″,6
Fourth - - - - -	24′ 4″

§ 114. If the simplest case of the mutual influences of the attraction in a secondary system, which we have treated in the third chapter, of the earth and moon, has already shewn us a considerable complication of effects, we certainly must expect still more in the case of the system of Jupiter, with four satellites so near to the great primary, that the nearest subtends an angle, or shews an apparent diameter, of about one degree at Jupiter, while the sun itself appears only under an angle of six minutes, that such a system must present

a great complication in the results of their mutual attraction, and, we might say, in some measure assimilate the orbits mutually to each other, under considerable oscillations. Indeed the investigation of this theory requires thirty-one different arbitrary quantities to be determined by successive suppositions, and approaches to truth; which is so much more difficult, as they represent to our observation only minute quantities, from which to deduce them leaves a considerable chance; as yet, they are sensible to us only by the retardation or acceleration of the eclipses. The simple statement of a few general results may suffice.

The inclinations of the planes of the orbits are not exactly constant, as we have already seen in the case of the moon, and they partake of the motion of the nodes of the equator of Jupiter, in his orbit, besides periodic variations, exactly as we have seen in the moon's orbit and that of our equator. To determine these quantities by observation is a matter of great nicety, or it may rather be considered as within the reach of theory alone.

The third and fourth satellites have two equations of the centre, the one depending upon the influence of the satellites upon each other, and the other on that of the planet. In the mean motions of the satellites, the following peculiarities are observed, viz: the time of the mean motion of the first satellite, and twice the time of the mean motion of the third, are together equal to three times the mean motion of the second. The mean longitude of the first satellite, and twice the mean motion of the third, less three times of the second, gives always two right angles. These relations are always rigorously true, and a mathematical consequence of the mutual influence of the four orbits upon each other; so that even the variations of long periods, which these orbits have like that of the moon, are subject to them.

The inequalities of the fourth satellite are as follows; that like the *annual equation*, depending on its mean anomaly, amounts to 1′ 53″,3; that similar to the *Evection* amounts to

21″,69; and that similar to the *Variation*, to 4″,21; in their greatest values.

§ 115. The revolutions of these four satellites around Jupiter, must evidently occasion the occurrence of phenomena analagous to what we have mentioned in the last chapter, as eclipses of the sun and moon at the earth; they occur so much the more frequently, as the body of Jupiter is greater, and the satellites are not only more numerous, but perform more rapid revolutions around it, in orbits but little inclined to that of their primary planet. At Jupiter, therefore, these phenomena must be of daily occurrence, under either the one or the other of the above forms; and opportunities for determining the longitude of places upon Jupiter's surface, by their means, must consequently be more numerous also, and more susceptible of accuracy.

To us, spectators from the earth, the observation of the passage of a satellite through the cone of the shadow of Jupiter, corresponding to our lunar eclipse, has been for a long time one of the most accurate means for the same purpose; it presents an instantaneous phenomenon, registered, as we have seen above, under different apparent times in different meridians, which has the advantage of not being under the influence of any parallax, as by Table II. line 26 and 27, that for Jupiter never exceeds 2″ in the arc, a quantity which, in this case, is entirely without influence; that it must be affected by the power of vision, that is, the good quality of the telescopes, and the state of the atmosphere, will appear from the simple reflection, that the gradual obscuration of the satellite, like that of the moon in its eclipses, will permit it to be seen longer by a greater power, when *it is immersed* in the cone of shadow, and earlier again when it goes out of it, or at what is called the *Emersion*. They are still of considerable interest, not only on this account, but in a general point of view.

§ 116. Plate V. figures 6 and 7 give the general representation of the circumstances of these phenomena, the first as

referred to the plane of Jupiter's orbit, or equator, or the planes of the revolution of the satellites, which, for this purpose it is not necesary to distinguish particularly; the latter to a plane perpendicular to this. The cone of shadow of Jupiter is of course always in a direction opposite to the sun, and is determined by the proportional magnitude of the sun and of Jupiter; it extends, in consequence, to about forty-seven times the distance of the fourth, or outermost satellite; the angle of its vertex is only 5′ 37″,5. The position of the satellite, in relation to this cone, in respect to its distance from the primary planet, and from the central line of the cone, evidently determines both the certainty and the duration of the eclipse, in conjunction with the velocity of the satellite in its revolution, and to the spectator on earth; it is also affected by the individual influences remarked in the preceding section.—As in the eclipses of the sun and moon upon earth, the latitude of the satellite, at the time of this passage, determines the chord of the circular section of the cone through which it will pass; the small inclination of the satellites to the orbit of Jupiter, and its considerable diameter, occasion that the two inner satellites, that is, the first and second, can never pass without being eclipsed; this has been indicated in figure 7, by the strong perpendicular line drawn in the shadow at the distances of these orbits; similar lines drawn at the distances of the third and fourth satellite, shew that the effect of the inclination of their orbits may bring them in the extreme position of their latitude to pass without being eclipsed; and from this, to the longest passage through the centre of the cone, all intermediate variations may take place, so that these two satellites, and particularly the fourth, has been observed to disappear only for a moment, or even only to become faint for one observer, while another did not lose sight of it; notwithstanding this, a mean duration, rather referring to the proportional velocity of the satellites in their orbit, and the diameter of the cone of shadow, has been admitted as follows:

		h.	m.	s.
For the First Satellite,	- - - - - -	1	7	52.
Second,	- - - - - - - -	1	26	3.
Third,	- - - - - - - -	1	46	50.
Fourth,	- - - - - - - -	2	22	55.

§ 117. In describing the synodic revolution of Jupiter, we have seen that near his superior passage, that is, when he appears to us on the same side as the sun, he becomes invisible, except with peculiar means, and under peculiar circumstances. For the same reason, the satellites and their eclipses must be invisible to us; the same must be the case when near the opposition of Jupiter, the cone of his shadow is hid from us behind the body of the planet. The more minute details will be evident from PLATE V., figure 6. The angle through the centre of Jupiter, on both sides of the axis of his cone of shadow, being made equal to that which we find in TABLE V., as the greatest elongation of the earth from Jupiter, viz: about 11° 5', and parallels to these lines being drawn touching the circumference of Jupiter, these will cut the cone of Jupiter's shadow and the orbits of the satellites, in such places as to shew the general conditions of the visibility of these eclipses. The part of the earth's orbit which falls in the cone of light between the Sun and Jupiter, which is the prolongation of the cone of Jupiter's shadow, is 26′ 2″ on each side of the axis of this cone, or the point of the opposition of Jupiter, and in this, therefore, the respective situation of the planets excludes the visibility of any eclipses. Within these two limits and their corresponding equal angles on the opposite side of Jupiter, the effect of the position of the earth upon the visibility of that part of the orbit of each satellite lying in the shadow, will easily be seen from the figure; for it is evident that the appearances will correspond to the angle which the visual line of the observer from the earth makes with the cone of Jupiter's shadow; the effect of this angle must of course be different for the different satellites, and this difference, as well as the details of the general pheno-

menon of each, will depend on the proportion of this angle and the angle under which Jupiter appears to the satellite.

The Earth, and Jupiter with his satellites, performing their revolutions in the same direction, when the earth, in her course, is approaching to Jupiter, and therefore towards the above-mentioned cone of light, as appears in the figure from the lower side, the entering of the satellites into the cone of shadow from the same side, will be observable by us; this motion of the earth corresponds in appearance to Jupiter's approach to his opposition, thence it is stated in astronomy, that, *before the opposition*, we see the *Immersion* of the satellites. The part of the earth's orbit within the cone of light, is passed by the earth in less than sixteen hours; after which the appearances change, the earth again receding from Jupiter, as in the figure, in the part above the cone of light; the other side of the shadow is now visible, which presents the re-appearances of the satellites; Jupiter appears to go again from opposition towards conjunction, and we say, *from the opposition to the conjunction* of Jupiter, we see *the Emersions* of the satellites. The intersections of the lines drawn tangent to Jupiter and the earth's orbit, with the sides of the cone of his shadow, show immediately, that, speaking in general terms of the first and second satellite, we can never see both the *Immersion* and the *Emersion* of the same eclipse; for, *before the opposition*, the body of Jupiter will hide the *Emersion*, and, *after the opposition*, it will hide the *Immersion*, as is indicated in the figure by the tangential lines cutting the respective opposite sides of the cone of shadow outside of the orbit of these two satellites. We may see both the immersion and emersion of the same eclipse of the two outer satellites, the third and fourth, when the earth is in a part of its orbit sufficiently distant from the cone of light and shadow, which we see decides so much of the appearances of these phenomena. These may even appear to take place at some distance from Jupiter, as follows from the larger range of the orbit of the fourth satellite, inter-

sected by the angle of the tangents and the sides of the cone of shadow.

The numerical value of these angles depends on the greatest elongation of the earth as seen from Jupiter, and the angle which he subtends for each of the satellites; and is easily ascertained from TABLE III. The first satellite seeing Jupiter under an angle of 19° 54′, does not, even in the extreme elongation of the earth of 11° 5′, admit of our seeing both sides of the cone of shadow at its orbit; the second, seeing Jupiter under an angle of 12° 58′, is in the same case; the third, from which Jupiter subtends an angle of 7° 55′, still leaves an angle equal to the excess of this, above 11° 5′, within which, counted from the point of greatest elongation, both *emersion* and *immersion* of the same eclipse will be visible from the earth; for the fourth satellite, whence Jupiter subtends an angle of no more than 4° 30′, this angle leaves an excess of 6° 35′, which evidently includes a part of the orbit so much the greater, as it is doubled by applying it to both sides from the point of greatest elongation, and as the motion of the earth is very much inclined towards its direction.

As we have always taken the full diameter of the section of the cone of Jupiter's shadow, as the object of comparison in these determinations, it will easily be conceived that these quantities are affected by the jovicentric latitude of the satellite, at the time of its transit through the shadow; this also affects the duration of the eclipses themselves, and they vary more in the case of the third and fourth satellite than in that of the second, of which latter, it is so very rare to see both immersion and emersion, that it is never calculated, and is considered, in our present state of information, as merely accidental.

§ 118. We have seen that the theory of the motions of the satellites of Jupiter is complicated, and that it requires a number of data to be determined from observation, which the small apparent magnitude they present to us renders difficult

to determine; and all this appears to be included within an angle of only 16′ 32″, which is all that the system of Jupiter subtends at the earth. All, therefore, that we can say of the eclipses of Jupiter's satellites has not that degree of accuracy which so many other parts of astronomy have obtained; still, in consequence of the use of these investigations in determining the times of the occurrence of eclipses, the determination of their epochs has much engrossed the attention of astronomers: the observations besides depend so much on the optical means employed in them, that they are not comparable with satisfaction.

However, an attentive observation of these eclipses has led to the discovery of a law in nature unsuspected before, and the effect of which had perplexed accurate observers in other branches of astronomy. The disagreements of the previous calculations with the actual observations showed a regular variation, increasing with the *distance of Jupiter from the earth;* this distance may evidently vary the whole diameter of the earth's orbit, that is between 4 and 6, nearly. The propagation of light had been considered as instantaneous for any distance, because its propagation through the short distances we can observe it on earth is not perceptble. On the contrary, applying to it the principles of all mechanical effects, that of succession and time needed to perform it, the inequalities showed themselves capable of computation by ascribing to light such a velocity as would enable it to pass the diameter of the earth's orbit in $16^{m.}$ $14^{s.}$ of time. The same result was almost simultaneously obtained from a discrepancy in the observations of the fixed stars, which produces a change of their apparent position, in a direction parallel to the ecliptic, of $40\frac{1}{2}''$, in the extreme of two opposite positions of the earth in her orbit; or of 20″,25, for the maximum of reduction to the centre of the orbit or the sun; this last phenomenon is called the *Aberration of light.* The discovery of these two phenomena of light concurred in giving to it a velocity of 10313 times as great as that of the earth in its

orbit. The influence of the progressive motion of light is now therefore well ascertained, and we shall have to speak of it in another place.

§ 119. The satellites of Jupiter are small in proportion to their primary, though they are larger than the moon, and some at a considerable distance from him, the farthest one in particular; hence, the phenomena which at Jupiter will produce an eclipse of the sun, by the passage of one or the other of the satellites before him, so as to hide it partially or totally from some point on the surface of Jupiter, though more frequent than the preceding ones, must remain unnoticed by us, with our present optical powers; for it is evident that they can present to us only a dark spot of very small size, or a mere penumbra upon the illuminated disk of Jupiter, of uncertain observation; it may, however, be reserved for posterity to become spectators of this phenomenon also.

Besides these phenomena, corresponding to our solar and lunar eclipses, Jupiter must still have those of the eclipses of his different moons by each other, which of course must be invisible to us. The occultations of fixed stars by the moons, must likewise evidently be more numerous; and the effect of the parallaxes of these moons, which are evidently the half of the apparent diameters of Jupiter, from each satellite, as given above, must introduce in these phenomena a still greater variation, in relation to the different localities, than that we observe upon earth; the same circumstances will also occasion the occultations of planets by one of the moons to be more frequent at one place than at another of the surface of Jupiter.

§ 120. Jupiter, with his satellites, exhibits to us a central body of considerable magnitude with smaller ones revolving around it; it therefore presents phenomena corresponding to those we have described above of Mercury and Venus in relation to the sun, namely, transits of the satellites before their primary planet, as seen from the position of the earth; over this also the passage of the satellites, in what we might

call their superior conjunction in relation to the earth, that is, when any one of them passes apparently behind the body of Jupiter itself, is visible to us; however the light of Jupiter is sufficiently strong to obliterate that of the satellite on its approach to the limb of the primary, so that these observations are but seldom very successful.

The time of any one of these phenomena, for any part of the earth may be considered as the same, because the parallax which, as we see TABLE II. line 26 and 27, is only between one and two seconds for Jupiter, may be considered as the same for the satellites, and is in this respect entirely without influence. A more cultivated theory of these satellites, and a still greater perfection in our optical means, which are both constantly in progress, give us reason to hope that all these phenomena will again become useful in the determination of the longitude of places upon earth. Now the observation still requires that the optical means employed by the observers should be equal, to render the observations comparable with a degree of accuracy any thing like corresponding to that of other observations of the same kind.

§ 121. *Saturn*, the next planet in our system as we recede from the sun, presents us with a system much more extensive than Jupiter; as he has *seven satellites*, and, moreover, presents us with a peculiarity entirely unique in the solar system, and an instance of the only form different from the spheroidic, in which matter free in space, subject to a revolution in a curve, and a rotatory movement upon an internal axis, can be in a state of equilibrium; namely, *a flat circular ring*, of small thickness, with breadth and extent appropriate to the mass. With all this train, Saturn performs his revolutions around the sun in the manner described in the first part of this book.

The disk itself of Saturn presents us some inequalities of shade and light, similar to all celestial bodies whose diameters we are able to observe; but the great distance of the planet rendering them rather faint, they have nothing par-

ticularly remarkable. In his greatest proximity to the earth, he is always more than four diameters of the earth's orbit from us; notwithstanding this, he appears with a light next in brilliancy to Jupiter. The apparent diameter of Saturn varies between 16″,3 and 20″,12; seen at the distance of the sun, it would subtend an angle of 2′ 52″, or about $\frac{1}{60}$ of that of the sun. The sun will appear to him under an angle of about $3\frac{1}{3}$ minutes; and to an observer in the sun he will appear, notwithstanding his ninefold distance, under a greater angle than the earth, namely, little less than 18″; his real diameter being nearly ten times that of the earth, all the other dimensions dependant on this follow, as in TABLE II. line 6 to 11. The mass which the most modern determinations give, of 115 times that of the earth, has long been considered as about 93 times that of the earth, and gives to Saturn a density of about $\frac{1}{9}$ that of the earth, as seen TABLE II. line 12 to 15.

The revolution of Saturn upon his axis, under an inclination to its orbit, of 21° 36′ 27″, takes place in about $10^{h.}$ $32^{m.}$ being, after Jupiter, the most rapid of all. The time of the revolution of his ring being sensibly the same, we may consider this ring as belonging to the planet itself, or at least as mechanically acting with it. The distance of the ring from the surface of Saturn is little more than the third part of the diameter of this planet; the breadth is about equal to the radius of the planet; more accurately, the outer diameter of the ring appears at a mean distance 38″,3, the breadth of it 5″,7, and it lies nearly in the plane of the Equator; thence it must act in augmentation of the flattening of Saturn, whether we consider it as making a part of his mass, or as exerting an extraneous attraction in the plane of the equator; we, in consequence, find this planet to have the greatest ellipticity or flattening at the poles of all, amounting to about $\frac{1}{11}$ of the radius.

§ 122. Some inequalities discovered in the ring of Saturn, have given the means for the determination of its revolution,

and, according to general principles, it may be expected to consist of parts like all the planets, the form of which in the nearer details must be determined by their gravity, and the facility of yielding mutually to each other, under the additional influence of the great centrifugal force, naturally created by the velocity of the rotation, and the free action of it upon a surface nearly parallel to its direction. With strong optical powers, this ring is found to consist of two concentric rings, a distinct dark line being discovered all around, nearly in the middle of its breadth. By the combination of the inclination of Saturn's ring to the orbit of the planet, and of this to the orbit of the earth, the ellipse which it presents to us becomes more or less elongated, and sometimes even presents to us no more than the edge, in which case it becomes totally invisible, with our present common powers of vision, notwithstanding the thickness of the ring is esteemed about equal to the diameter of the earth. By the same change of position, the two sides of the ring are illuminated alternately, and we may, in consequence of this, again lose sight of the ring, by our viewing it from the dark side, when it will present itself merely as a dark belt over the planet, or also when only the edge of the ring being illuminated by the sun, our optical powers do not suffice to discover it; that is, when the protracted plane of the ring passes through the sun, which happens at $5^s\ 20^\circ\ 53'$, and $11^s\ 20^\circ\ 53'$; then again the dark part shows upon the body of Saturn like a shadow; shortly before and after that phenomenon, we also see the actual shadow of the ring upon the planet. The northern surface of the ring will be illuminated while Saturn is between $5^s\ 20^\circ\ 53'$ and $11^s\ 20^\circ\ 53'$ of his heliocentric longitude, and on the southern side when the planet is in the opposite part of his orbit. The ascending node of the ring was, in 1801, in $5^s\ 17^\circ\ 30'$, and it has a motion in the order of the signs.— The most advantageous views, or the invisibility of the ring, alternate in the present times as follows, viz:

1825, in November, south side best illuminated.
1833, in April, ring invisible.
1838, in July, north side best illuminated.
1847, in December, ring invisible.
1855, in April, south side best illuminated.

§ 123. We know nothing positive in respect to the atmosphere of Saturn; very minute observations have of course great difficulties at that great distance, but we cannot refuse to ascribe one to him from analogy to the earth and other planets, and from the general principles; the peculiarity of having a ring, must occasion peculiar modifications in this respect, as we would expect to take place at the earth, if we were to suppose a similar ring to surround us, at little more than one diameter of the earth from its centre; for according to TABLE III., it will be observed, that its distance is only 1,166 diameters of Saturn, or $12\frac{2}{3}$ diameters of the earth.—From this ring, the diameter of Saturn subtending an angle of more than 46°, it is evident that the appearance of this planet, always 23° above the horizon of the ring, on either side, must present a spectacle entirely different from any which the astronomy of the earth presents to us. If we might venture into the field of speculation in relation to the possible improvements of the state of the intellectual inhabitants of Saturn, we might dare to promise to them, from the invention of aerostats, the advantage and pleasure of a communication between the ring and the primary planet—a thing impossible between two spheric and distinct celestial bodies, having separate spheres of attraction; a peculiar modification of the atmosphere of the planet between the ring, comparatively to that on the other part of the planet, must of course exist, under the influence of this combined attraction. The radius of the earth's orbit subtending an angle of only about 6° at Saturn's orbit, and this angle, as we have seen above, being equal to the part of the apparent disk, or the angle at the centre of the planet, which we might, in the extreme case see in darkness, we are not able to distinguish any

change, or what is usually called phase, of the planet. But the great variation of the aspect of its flattening, joined to the variable inclination of the polar axis towards us, of upwards of 30° on each side, produces a greater and very observable change of his disk, and we see it in full only when we are in the protracted plane of its equator.

§ 124. In PLATE III. the figures at the right side show the system of Saturn, together with that of Uranus, in their comparative magnitude with the sun, the upper figure presenting the aspect of the system from the earth, under the most favourable inclination, the lower figure containing a semi-circumference of the sun upon half the scale of that of the figure for the system of Jupiter and the earth, it has also the two planets and Saturn's ring for the central figures, as the other has Jupiter and the earth. The section perpendicular to this plane, generally taken for that of the satellites, is represented by the several lines passing through the planet, denoting the orbits of the satellites, under their proper inclination, together with the section of the ring on each side.

§ 125. Saturn has seven satellites; their whole system subtends to our view an angle of no more than 17′, 25″, and its extreme radius is only 0,0242 of that of the earth; these satellites appear to us under an angle of less than one second, though in reality they must be larger than the moon, in order to become visible at the great distance which we are from them; being, therefore, observable only with the best optical means, there is no probability of their furnishing us means of practical applications like those of Jupiter, by their eclipses. While Jupiter's satellites were discovered almost as soon as telescopes themselves, the discovery of those of Saturn is of a much later date, and due to a very advanced state of improvement. Their proportional distances and times of revolution, are as yet all that we know of them; the former are represented in PLATE III. in the figures above quoted, and the latter are apparent from TABLE III. under their proper heads; they vary from about $22\frac{1}{2}$ hours, for the

innermost satellite, to $79\frac{1}{3}$ days, for the seventh or outermost one. The six first satellites lie nearly in the same plane with the ring. It is supposed, both from their appearances, and their analogy to the moon, that they always turn the same side towards their primary planet. In like manner, the laws discovered by Kepler, to which we know that the satellites as well as the primary planets are subject, render it certain that their orbits are elliptic, and not circular; still we have as yet only a determination of the ellipticity of the orbit of the sixth satellite of 0,048876, as seen in TABLE II. The probable smallness of the ellipticity of the inner satellites, the inclination under which we see their orbits, and their great distance, all concur in rendering this determination difficult.

Theoretical investigations have shown that here, as in the case of the satellites of Jupiter, a constant mean plane exists, having the node of the equator of the planet and the satellites constant. This has generally been found a phenomenon coinciding the more accurately with observation, as the distances of the satellites were smaller; similar investigations also show, that the orbits of the satellites are the more inclined, as they are more distant from their primary.

§ 126. These satellites must of course present the same phenomena of *Eclipses* of the *sun* and *moons* to their primary planet, as we have seen occurring in the case of the earth and Jupiter; they will also exhibit to us phenomena similar to the eclipses of Jupiter's satellites in the cone of the shadow of their primary; but they must evidently be much less frequent than in the case of Jupiter, on account of the great inclination of the orbits, which must limit their occurrence to the time of the passages happening very near the nodes as in the case of the moon and the earth. Besides, the greatest digression of the Earth from the Sun as seen from Saturn, is only about 6°, as given in TABLE V.; hence, these phenomena must take place for the greatest part of the time without being visible to us; and in all cases, the eclipses of the satellite

must appear to happen so near to the body of Saturn as to be unobservable, on account of his preponderating light. The cone of the shadow of Saturn will extend as far as about 38,6 times the distance of the farthest or seventh satellite, or 4578 diameters of the earth, the angle of the vertex of the cone will be only 17¾ seconds. We seldom see even the shadow of Saturn upon the ring, for the same reason, of our too small deviation from the straight line between the Sun and Saturn. For the planet itself, this ring must occasion the most peculiar phenomena, by the constant and slowly moving shadow which it constantly maintains upon some part of the planet, and the changing appearance or disappearance of the satellites behind it, as seen from the one or the other hemisphere of the planet.

§ 127. *Uranus*, the farthest planet of our solar system, although his diameter is more than four times that of the earth, appears to the naked eye as a star of the fifth magnitude. He, in consequence, continued for more than a century after the discovery of the Telescope unknown to us. This arose from these instruments being yet too feeble to enable observers to detect an actual determined disk, which could have called the attention of astronomers to his motion. His distance from the Sun is nineteen radii of the Earth's orbits, and, therefore, his distance from the Earth varies no more than $\frac{1}{9}$ of the mean. This, joined to his small apparent diameter, makes the variation in his visible magnitude hardly sensible. He appears to subtend the angles of 4″,1 and 3″,7 in the two extreme distances; at the distance of the sun, he would appear to us under an angle of 1′ 14″,5, and he sees the sun nearly under the same angle, namely 1′ 18″,6; the angle which he subtends from the sun is, of course, nearly a mean of the two extremes of his appearance from the earth, or 3″,84. As the earth always subtends an angle of less than 1″ at this planet, or what is the same thing, the parallax varying only between 0″,47 and 0″,42, it was so much easier for us to consider it as a fixed star, while for him the earth

dwindles to about the same diminutive angle under which we see the satellites of Jupiter and Saturn. The latest determination of the mass of Uranus, gives him 22,65 times the mass of the earth, it was before generally received as about 17 times; the density becomes by it nearly $\frac{28}{100}$ of that of the earth, and the specific gravity of course proportional, both being little more than that of the sun, notwithstanding the planet is rather one of the smaller; but it must be observed, that all data referring to these minutia of this planet may yet receive considerable variations from longer-continued observations. The other details may easily be followed in TABLE II. The line of his rotation has not yet been determined directly, because at his great distance, within such a small angle as he subtends to us, the observation of any spot becomes nearly impossible; but if we are authorised to conclude by analogy from the other planets, that the plane of the equator of Uranus and that of the orbits of the Satellites are not much inclined to each other, and that the axis of his poles remains approximately parallel to itself during the whole revolution, this axis would only make an angle of about half a degree, and therefore the variations of the seasons be of such immense difference from ours, as we might easily conclude by supposing that the neighbourhood of our poles of the earth, instead of being deprived of the sun only for about from three to six months, were so for the same number of years. The Torrid Zone is only about 1° in breadth, and the position of it changes from a direct exposure to the sun to that of having the sun only under a small angle over the horizon during a number of revolutions, a phenomenon which we know to be peculiar to the neighbourhood of our polar circles; every part of Uranus would see or lose the sun for more than one rotation, or as we would say in usual language on earth, would have days or nights of more than once twenty-four hours.

§ 128. According to the latest accounts, Uranus has *seven satellites*, but we have as yet only the orbits of six determined,

the known elements and data of which are calculated under their proper head, in TABLE III. The determinations which we have in respect to these, are still principally due to the discoverer of the planet himself, namely, Herschel; the second and fourth however have been more accurately determined by other astronomers. The radius of the whole system of these six satellites is about $\frac{1}{60}$ of that of the earth's orbit, and it subtends to us an angle of only about five and three-quarter minutes. Its magnitude, compared with the sun and the system of Saturn, is elucidated by the figure on PLATE III. The great inclination of the orbits of these satellites of 89° 30′ to the plane of the orbit of Uranus, forms their most remarkable feature, while the greatest inclination of those of other planets, namely those of Saturn, amounts only to about 25°. These orbits, we know from theory, must be elliptic; but as yet only the eccentricity of the sixth has been attempted to be stated at $\frac{4}{10}$, which would be by far the greatest that we know of.

The eclipses of these satellites, or the solar eclipses which they must occasion upon their primary planet, must have the peculiarity of being confined to two certain epochs in opposite parts of the year, while, in the other parts of it, they must be entirely impossible.

We have still, in the present state of our knowledge of this planet and his satellites, much to desire, and, as there is every reason to hope, also to expect, from the future improvements in astronomy, particularly the daily improving art of increasing our optical power, by which we may be enabled to penetrate into the details yet remaining undistinguishable by us at such great distances.

CHAPTER VI.

Of the Comets.

§ 129. As we propose to treat in this third part principally of the more minute details and physical constitution of the celestial bodies, this chapter upon the comets cannot afford as satisfactory results as we have been enabled to present in the preceding chapters. Our knowledge of the comets is still limited almost entirely to that of their course, and the elements of their orbits. Both these appear very variable, and have been already treated of in the fourth chapter of the first part; observations upon their physical nature must of course be very imperfect, on account of the short time during which they are visible to us, and the powerful optical means which they require.

We may suspect, from what has been seen of them in general, that those comets that appear the most brilliant, are not those that would in reality be the most important; since comets apparently small have been discovered, whose orbits being more closely connected with our planetary system, we may suppose them also in their mass, density, and nature nearer to the planets; however, their mass, concluded from their want of influence upon the orbits of the neighbouring planets, is inconsiderable.

Their *appearance* as mere masses of light, and transparent, as we know light to be, may in future times instruct us as to the nature of light itself. Their permanency, now averred

by the accurately known orbits of a certain number, and the certainty of their return within our view, in bringing them under the regular action of the attraction of the sun, and consequently also of the planets, (the last of which has been calculated in various remarkable instances,) has not yet presented us any reaction on their part; their mass, therefore, is exceedingly small; thus, for instance, in the comet of 1770, the attraction of Jupiter in the neighbourhood of which it must have passed in 1767, altered its orbit so materially as not to make it appear the same comet; while on the contrary, notwithstanding it came the nearest of any comet to the earth, its attraction upon us was entirely insensible. Under the supposition of an equality in the mass of the two bodies, according to the laws of attraction, the effect of the comet upon the earth would have been an increase of the sideral revolution of the earth of $1\frac{1}{6}$ day; observations and the solar tables, however, concur in proving that no alteration has taken place amounting to as much as two seconds in time, which reduces the possible supposition of the mass of the comet below $\frac{1}{5000}$ of that of the earth. If then, notwithstanding the large diameters under which comets are occasionally seen, their mass is inappreciable, it must be impossible for us to determine their densities.

The physical constitution of the comets thus appears to constitute in the great scale of the universe a link between the solid masses of matter and the ethereal or aëriform, and this as *subject to universal gravitation* as we find all that we can call matter on our globe. But our imagination, closely bound to material subjects in nature, furnishes us with no type upon which we might imagine animated beings to people the comets, and to enjoy the spectacle of the varied scenery in what we might call a voyage through our solar system.

The most remarkable Comets which struck the ancients, before they were considered as subjects of astronomy, while the apparently smaller and, in astronomy, more interesting ones remained still unobserved, are often accompanied by a

luminous train, which occasioned the denomination of comets, or stellæ comatæ, being given to these celestial bodies, from the appearance which they presented. It was formerly supposed that this phenomenon only occurred on their going away again from the proximity of the sun, but this has been found as little constant as any of the other details accompanying their phenomena; this appearance, usually denominated the tail, often extends through such an immense space as to subtend to our view an angle of 20°, or 50°, and its appearance is usually under a constant succession of changes. This train is often double, or even triple, variously inclined and curved. We have, however, so little that is interesting to say of its nature, or of the causes for its shape, that the inquiry appears to appertain rather to the Physics, than to the Mechanicsof the heavens.

§ 130. Old established superstitions, prejudices, proverbial expressions, and old sayings, or figurative expressions, have most generally their origin in some fact, or accidental coincidence of circumstances, which made a lively impression upon the untutored mind of man, and for which his natural propensity to inquiry led him to seek an explanation, or a connexion, which imagination supplied, once happily, once in direct violation of sound reasoning.

The extraordinary and unforetold appearance of a celestial body of uncommon form, could not fail to be associated with some nearly simultaneous unforeseen event, which appeared to be out of the common order of occurrences; slight efforts of designing men were sufficient to give consistency and permanency to such an idea, in times when mathematical science and such events were not suspected susceptible of connexion. The greater pretended wisdom of astrology, which accounted for the combination of the common events of life, by the combination of the position of the planets at the moment of its beginning, could not fail to maintain also the connection of the extraordinary appearance of a comet with the extraordinary events of the moral world;

and the greatest gain of the designing being always in acting upon the fear of the weak, prognosticating harm from an uncommon phenomenon must of course obtain the preference over the holding forth the hopes of good.

This may have been the history of the establishment of the fears of the ignorant upon the apparition of comets. But the fears appeased by the light of science, were not suffered to die away entirely, and if *certainty* failed, at least *probability* or possibility was yet reserved to maintain them. The courses of the comets are various, subject to great variation, and distributed apparently without law; their number appears as yet unlimited; hardly a year passes that one or two are not discovered. Such a traveller from a great distance, of an unknown mass, might in his course strike the earth! and, if delivered from the fear of the partial misfortune threatened by astrology, it was attempted to frighten us into that of the universal destruction of the whole earth; and, even if that should not be the consequence, still we know that the approach of two celestial bodies must proportionally increase their attraction; hence a proportional increase of danger must result for fearful man, of being struck off from his abode, or drowned by the seas overflowing under this extraordinary attraction; our climate should be subverted; or—; but who can calculate all the misfortunes which imagination can create? We are not even at a loss to find in the ancient words of tradition, accounts of events corresponding to the theoretical result of the effects of gravitation in such a case.

§. 131. But as little as we may hope to change the laws of nature by our fears or deprecations, so much may we calm those fears by cool calculation, and a few sound reflections, supported by the consciousness of moral rectitude, which shields from all apprehensions for futurity; *he* only needs be fearful who is conscious of deserving no good fate; without such feelings, man may await the results of the laws of nature with full confidence, equable mind, and calmness. The pro-

bability of the chance of the meeting of two celestial bodies, even in the case of their orbits crossing each other, is exceedingly small, in respect to space, on account of their diminutive size in proportion to the space which is given to them to move in, the earth occupying an arc of no more than 17" of its own orbit; not only a simultaneously equal distance of the comet and the planet from the sun is required, but also the same heliocentric longitude, and that the disturbing visitor should have no latitude, or at least only a few seconds. The great velocity of the comets, particularly near the perihelion, where alone such an event could take place, limits this event to the short space of time, during which they would describe the small arc within which any effect *could* take place; thus, for instance, the comet that has come nearest to the earth, would have passed the diameter of the sphere of his influence upon the earth in about twenty minutes of time. That from the greatest number of comets no apprehension at all need be felt, because their orbits lie entirely, and considerably, outside of that of the earth, is besides evident by itself.

The *Mass* of the comets is so small that they could exercise but a trifling and even local influence upon the body of any of the larger planets, and particularly upon the dense earth; so that almost close contact would be required to produce any effect at all upon it. Notwithstanding the comets of short periods lately discovered revolve at no great distance from the planets, not having been known, their influence was not accounted for in the perturbations of these planets; still the motions of these planets have been so fully accounted for, in conformity with the theory, from the attraction of the other planets alone, that the attractive influence of the comets may be considered as proved to be entirely ineffectual.

All this may sufficiently prove that the disastrous phenomenon of the disturbance of the earth by a comet, though not *absolutely impossible* by the laws of nature, and which may perhaps have happened about five or six thousand years ago,

is devoid of all probability for a great multitude of such periods; during which the human species may continue to strive, during many long series of successive generations, to improve its moral and intellectual worth and happiness, free from all possible fear on that side.

CHAPTER VII.

General Considerations in relation to the Solar System.

§ 132. After having surveyed the details of the planets and comets of our solar system, we may with propriety take a general retrospective glance, in order to compare the facts observed, and the results obtained, and to give a general idea of the principles by which the mutual influence of these masses upon their respective motions are founded.

What must first strike the imagination, in this general view, is the immense overpowering influence of the sun, extending its attraction to a distance where its diameter subtends an angle no greater than about 1′ 18″,6, as is the case at *Uranus*; and even farther, if we are in any way near the truth in the dimensions of the orbits of so many comets, whose semi-axes even far outreach the distance of this farthest known planet of our system.

This naturally leads us to compare it in all respects, as to dimension, volume, mass, &c., with the sum of the like quantity in all the planets. This comparison is easily obtained from Table II., which furnishes us with the following remarkable approximate results: viz.

The sum of all the Diameters of the planets is about	$\frac{10}{38}$	of the sun's diameter.
Surfaces - - - - - - - -	$\frac{1}{55}$	- - - - surface.
Volumes - - - - - - - -	$\frac{1}{596}$	- - - volumen.
Masses - - - - - - - -	$\frac{1}{781}$	- - - mass.

To the sum of the masses of the planets the satellites have been added, estimated approximately as equivalent to

twenty-five of our moon's masses, although it may be said that these are already included in the estimates of the masses of the planets, deduced from their influence in the perturbations they occasion in other planets.

In the sum total of all these magnitudes, together with that of the sun, the share of the earth is of course small; it presents us with the following fractional magnitudes.

Of the sum of the Diameters, that of the earth is about	1/141
Surfaces - - - - - - - -	1/12713
Volumes - - - - - - - -	1/1807[illegible]
Masses - - - - - - - -	1/350081

In this comparison, it is remarkable how much the great volume of the sun, comparatively to its density, influences the result; the sum of the volumes, compared to that of the masses, making the latter fraction more than four times that of the former, on account of the much greater density of the earth.

The fourth PLATE represents, upon a proportional scale, the disk and diameters of the planets, and a part of that of the sun; the inclined lines that are drawn through the centres, represent the axes of rotation with their inclination towards the orbits of the planets, and, in respect to the sun, with our ecliptic.

The preceding chapter has given the reasons why the comets cannot enter into this comparison, their diameters and masses not being ascertainable by our means.

§ 133. According to the laws of universal gravitation, any two celestial bodies, connected in their revolution, must describe ellipses around their common centre of gravity similar to each other, and in the magnitude of their lineal dimensions inversely proportional to their masses; therefore, their mechanical effect is exactly analogous to that of two weights on a lever. The deviations from this result which are observed in our solar system, are due to the effect of the same law of gravitation, exercised mutually upon each other by

the planets which revolve around the same counterpoising central body, the sun; these effects are called *Perturbations*.

If we reduce, by a simple approximate calculation, the whole mechanical effect of the planets, counterbalanced by the [illegible], to its simplest possible case, according to the above view, which however can never occur in nature, namely, a general conjunction of all the planets on the same side of the sun, we obtain a lever, on the one arm of which the sun, and on the other all the planets are placed, each at its respective mean distance; for this extreme case the common centre of gravity is found at less than two diameters of the sun from its centre. The planets being always distributed in various directions about the sun, this common centre of gravity, at which the sun and the planets must always be in equilibrio, must be subject to a change of distance from the centre of the sun, corresponding to this joint effect; unless these planetary motions be so combined as to maintain, by their joint effect, the centre of gravity of the system always in the centre of the sun itself. To decide whether this is actually the case in nature or not, is a task for mathematical analysis which might be of interest, because, if by this result the sun should be subject to any motion, as we have seen to be the case between the earth and moon, it must naturally have an effect upon the apparent position of the fixed stars, which we can determine by no other means but by reference to the ecliptic, the centre of which would then become variable; for we must consider the reduction or reference to the plane of the equator as a mere individuality for our convenience, on account of its coincidence with the daily rotatory motion of the earth. This cause might occasion a part of the small apparent motions of the fixed stars. It is, however, evident from the small distance to be distributed among the variously combined mechanical effects of the planets, that the common centre of gravity of the system will always lie within the sun itself, as we have already seen to be the case with the earth

and the moon, and as probably may be still more the case with the other planets having satellites.

§ 134. The combination of the laws of gravity expressed in the beginning, show that *the square of the times of the revolutions of the planets are directly as the cubes of half the great axes of the orbits, and inversely as the sum of the masses of the sun and the planet*, leaving all external influences or perturbations out of consideration. The overpowering mass of the sun, as compared with any one of the planets singly, renders the addition of the mass of the planet of no influence, and substitutes, in the inverse ratio, merely the mass of the sun. When both masses are moveable, the relative gravitation of the smaller to the greater is as their sum is to the greater.

In the orbits of the comets, which are still calculated as of a parabolic form on account of their great eccentricity, the lineal dimension to which the above proportion is referred, is the parameter, which is the diameter perpendicular to the axis, and passing through the focus of the parabola.

The measure of the attracting force of a celestial body, exercised upon another revolving around it, is given by the velocity which it communicates to the latter; this is in the ratio of the mass. The influence of another body, external to these two, is the remaining effect of that velocity, which this body would have impressed upon it, in the absence of the overpowering influence of the other. Therefore, when the planet has a satellite, this comparison may be made directly by the comparison of its time of revolution with that of the planet around the sun; in this way the relative masses of the three bodies can be determined. When the mass of one planet is known, then its comparison with the sun and another planet gives the means to determine the mass of this other.

§ 135. The determinations of the masses of the planets, and of their mutual influence upon their respective revolutions which form the *Perturbations*, are therefore intimately connected, and dependant upon each other; the number and

variety of these influences render the mathematical problem of the perturbations very complicated, so that an *accurate* and *general solution* is still beyond the reach of mathematical analysis. The solution, in the case of the solar system, is obtained by an approximation, favoured by the small eccentricities and mutual inclinations of the planetary orbits, and the small masses of the planets compared with the mass of the sun; and even with this limitation it still requires the most subtile and accurate mathematical analysis. More, therefore, than a general idea of the elements, and some of the remarkable results is not to be expected here. These *Perturbations* do not depend alone upon the gross influence of the relative situations of the planets in their orbits, considered as regular, and the planets as revolving regularly in them; the mutual attraction of the heavenly bodies is also, in this case, as in the general phenomena of their motions, the only vital power, and the law of its action is the same; the different directions in which it acts, and the different distances at which it exercises this action, are so many variable causes of discrepancy in the resulting effect; the whole of these calculations, therefore, refer to the relative geometric position of the acting bodies and their power of action by virtue of their mass, exactly as has been seen in the case of the moon. On the combination of these elements, therefore, depend the variations of the influences, and their ultimate effect at any given time. The variations which the elements of the orbits themselves suffer, are sufficiently great to influence these quantities; this again introduces a greater number of variable and variously combined elements into the calculation, the effect of which becomes more or less sensible, and returns more or less rapidly to the production of the same quantity in the result. In this way, periods of different lengths, that regulate the calculations, are formed.

The *Periodic Perturbations* depend on the situation of the planets in their orbits in respect to the nodes, or the greater axis of their orbit. The *Secular Perturbations* depend on the

gradual variation of the position and elements of these orbits themselves, their smaller magnitude renders them sensible only after a greater lapse of time; a regard for the ease of calculation has alone introduced the use of the centenary periods, with which they are no more connected than the life of a man is with the year by which he counts its length.— The principal results of these secular variations of the elements of the planetary orbits, are contained in TABLE I. under their proper heads, and with the arithmetical signs +, or —, denoting, the first an advance, the second a retrogradation, in the regular order of the signs of the ecliptic, or an augmentation or diminution of latitude.

§ 135. Among all these influences, those that have the most general and the most sensible effect upon the appearances, must evidently be such as affect the position of the earth itself, and its orbit; they act in the manner we have seen in speaking of the attraction between the earth and the moon, to displace the observer himself, and consequently occasion a variability in the aspect of the whole universe. The influences depending on the masses of the largest planets, Jupiter and Saturn, have first been accounted for, both on account of their more sensible magnitude, and the better determinations of the masses of these planets; in respect to the earth, Venus and Jupiter were first accounted for, as most influential; Saturn, Mars, and Uranus have been taken more lately into consideration. Mercury appears to have the least influence of that kind, acting, to all appearance, conjointly with the sun; besides, uncertainty still exists in respect to his mass, and causes an equal uncertainty in all the calculations depending on it. Amidst all these variable elements of the orbits of the planets, the sum of which always produces a perfect balance in the whole of the system, the mean motions of the planets in their orbits have been found constant; which may corroborate the idea, that the centre of gravity of the system may also be constant in the centre of the sun; while all the other elements of the orbits are subject to an augmen-

tation or a diminution, alternating like an oscillatory motion. This constancy proves that, whatever may be the masses of the planets, still in consequence of their motion in the same direction, in orbits not much inclined to each other, the whole system oscillates about a certain mean state; that these oscillations are confined within certain limits; that no planet can ever have been a comet, nor alternately; even the circumstance of the apparently constant diminution of the obliquity of the ecliptic, so important to us, although so minute in respect to the whole system, oscillates within certain limits, and according to theory, can never sink below 20°, nor exceed 28°; for the determination of its epoch, the whole history of astronomy has as yet furnished us with too small a portion of the diminution, to allow us to pretend to a determination of it from observation; the oldest observations entitled to credit that are known, show it to have been about 23° 54′, about one hundred and six years before our era, and at the beginning of the present century it was 23° 27′ 57″, giving for nineteen centuries a diminution of only 26′. No fear therefore need be entertained of the evil consequences of a coincidence of the planes of the equator and the ecliptic.

§ 136. We have still, in treating of the effects of the perturbations, to make, as Copernicus did in forming a true representation of the Solar System, certain assumptions of some of the elements. These ought of course to constitute a system that will represent the actual state of the appearances. But when a difference between them is observed, we are provided with the data for an inductive process, that may be applied in inverse order to correct these assumptions. We thus obtain what may be called an oscillatory approximation, with diminishing variations. By theory and observation, then, stepping alternately forwards, it has, especially in the later advancements of astronomy, been proved that the application of the exact sciences to nature, will afford a benefit in a ratio like that of compound interest; every improvement, or advance, becoming productive as

a new principal; for such is the progress made, that, in most cases of importance, a single second in time, which in common life is wasted and abused by millions, becomes a quantity claiming the attention of the calculating astronomer; and even a subdivision in determining the place of a heavenly body, or the time of certain phenomena. A second in the arch, a quantity entirely microscopic to the human eye, and undeterminable in common life, is by no means neglected; thus for instance the tables of the sun, that is to say, of the motion of the earth in its orbit, made by several late astronomers, give results not differing, at a mean rate, so much as these small quantities, either between themselves, nor from nature and the actual appearances; accuracy approaching nearly to this, may perhaps be claimed for the tables of the moon, although this body presents us with the greatest angular variations for every trifling influence, whether simple or combined. The tables of the revolutions of the planets, the results of zeal for the science alone, without the particular interest attached to those of the sun and moon, present results that are in proportion no less satisfactory; for in these the direct interest, which has prompted governments to call for the perfection of the tables of the sun and moon by pecuniary rewards, does not as yet exist in the present state of civilization.

The want of accuracy in the ancient observations of the planets, and the shortness of the time whence we date accurate modern observations, has of course retarded the perfection of these tables. In the minute details relating to the planets themselves, and the perturbations of long periods, the simultaneous perfection of mathematical theories and of astronomical instruments, the means and methods of observation and calculation which skill and zeal for this science have of late produced, will probably soon bring these parts of the science to a satisfactory result.

The two largest planets, Jupiter and Saturn, having been the longest observed, give also the best results, and therefore

furnish tests for the theory of universal attraction in its details, which they confirm in every part. By the late discoveries of the small planets, *Vesta*, *Juno*, *Ceres*, and *Pallas*, between Mars and Jupiter, and the *Comets of short periods* in their neighbourhood, opportunities will be presented to improve the details of the theory of attraction, the effects of which they will present under more varied circumstances, in consequence of the greater eccentricity and inclination of their orbits, their intermixed course, and the great disparity between their masses and those of their neighbours, particularly Jupiter, the greatest of our whole system, which has, for instance, already been found to produce, in the motion of Ceres, perturbations that may amount to and vary from $3\frac{1}{3}$ minutes to 10 minutes.

§ 137. It may be proper here to make a collection of data and results, which are of interest and proper to be stated, in the same manner as we have done in respect to the moon. The sum of all the quotients resulting from the division of the mass of a planet by its great axis, (or considered as that of a variable ellipse,) is always very nearly a constant quantity. The secular variation of the equation of the centre of the planets, is the result of the mutual influence of the planets upon each other's orbits; and such are in general the causes of all the variations recorded as secular.

Retrospective calculations of the variations of the position of the orbit of the earth, show, that 1248 years before our era, its greater axis was at right angles with the first point of Aries, and that 2841 years earlier, this axis coincided with the same point, this epoch is about that at which chronologists have placed the creation of the world.

A peculiarity in the planets, analagous to what we have observed in the satellites of Jupiter, is, that five times the mean motion of Saturn is equal to three times the mean motion of Jupiter; this occasions a periodic coincidence, that depends upon a more general equation of a long period.

The inclinations of all the axes of rotation of the planets

remain *nearly* parallel to themselves, and form the same angle with their orbits, during their revolution around the sun; on this depends the constancy of the seasons, which we can readily conclude from this to be very different in the different planets. The changes of this inclination are most probably subject to the smaller or larger variations in all the planets, analagous to what we observe on earth, and probably more varied in the larger planets, under the influence of more satellites and a greater flattening at the poles.

According to theory, to each velocity of rotation correspond two different ellipticities; hence the determination of the one existing depends on observation.

The planes of the orbits of the satellites lie always between the plane of the orbit of their primary planet and its equator; in Saturn, only the plane of the seventh satellite deviates much from that of the ring, the others are all nearly in the same plane with it, having their nodes common with the equator of the planet; in general, the inclinations of the satellites' orbits increase with the distance.

§ 138. If we were to consider heat and light as a material emanation from the sun, as has been usual, we would calculate and determine upon mathematical principles the proportional quantity which each planet would have, as has been attempted. But the futility of this reasoning appears palpable, for the simple reason, that the results appear not to answer any purpose in nature adequate to the effect which we see. In treating of the sun, we have proposed a theory which appears to fulfil the demands of nature upon principles of sound philosophy, and of which we have found confirmations in the appearances and the proportional densities of the planets. All mechanical effects of inanimate nature are exercised in straight lines, in which also the action of heat and light is directed, and we can pursue and study their mechanics or principles of motion, by the same methods that we employ in the cases of gravity. Upon the different capacities of different kinds of matter, of retaining or communicating

heat and light, we have examples on a small scale on earth, but cannot as yet, by any direct conclusion, apply them to the celestial bodies, so as to determine absolute quantities of it, although the general principles stated in the earlier part of this work have acquired confirmation by a comparison with the results presented by the planets.

One difficulty we still find in making the density of the sun and the larger planets coincide with what we know or observe on earth, in regard to the solidity of the matter composing these large bodies. With such a small density and specific gravity as we have indicated, and as has been generally adopted, we find no indication of that solidity which, in our experience, we must consider as indispensable to constitute a celestial body; the specific gravity of resinous substances presents us with a solidity only feeble in low temperatures, and with complete softness at the higher temperatures of our atmosphere, while we must, upon all grounds, allow to the sun the highest temperature, in order that he may occupy the station, and fulfil the functions, we see he does in our system. But all these difficulties cease, on returning, in our inquiries, to the origin of these numerical determinations. Ingenious and careful as the operation for determining the specific gravity of the earth by means of the measurement of the mountain of Shelhalion was, it rests upon the *mere supposition* of its density, relative to that of the earth, and we may with much more propriety refuse to this supposition the necessary accuracy to make it a basis for conclusions so extensive, than we may admit a result contradictory to what we otherwise find as a general principle. No doubt, new inquiries, by different means and methods, will arise in the present state of science, when so great an activity exists in all researches that may enlighten and extend our knowledge of the earth. Already, pendulum observations, made twelve hundred and twenty feet deep in a mine, have shown a variation from corresponding ones made at the surface, which lead in result to a density of the earth of 7,73 times that of the surface,

or about twenty times that of water, instead of 4,715, as we have stated before upon the habitual indications; the difference between these oscillations having been 8″,23 instead of 2″,46 which would follow from the former determination.

We may therefore expect that this difficulty will be solved by future experiments and observations; and it will easily be observed, that the proportions of the masses to that of the earth, taken as unity, being all that is used in astronomy, the results obtained by the former supposition will not suffer any change, but only re-act upon our knowledge of the physical state of the earth, and the comparison of it, and the different kinds of matter we find upon it, with the other bodies of our solar system.

As for the proportion between specific and sensible heat, upon which we might consider the question of habitability to depend, according to our supposition of the state and wants of intellectual beings, it is as yet unknown, as much as that of the medium in which they live corresponding to our atmosphere; therefore, we may with propriety not consider ourselves as furnished with sufficiently accurate data on either of these questions to do more than simply acknowledge that sufficient general data are also sufficiently convincing indications of the general inhabitation of all the planets, though we are, as yet, and shall ever be, unable to determine the respective properties of both, the state of the surfaces of the planets, and of that of the inhabitants, that is, the intellectual beings dwelling upon them.

or about twenty times that of water, instead of 4,71.., as we have stated before upon the habitual indications; the difference between the oscillations having been 0.23 instead of 2.., in which would follow from the former determination.

We may therefore expect that this difficulty will be solved by future experiments and observations; and it will easily be observed, that the proportions of the masses to that of the earth taken as unity, being all that is used in astronomy, the results obtained by the former supposition will not suffer any change, but only re-act upon our knowledge of the physical state of the earth, and the comparison of it, and the different kinds of matter we find upon it, with the other bodies of our solar system.

3. For the proportion between [illegible] and sensible heat, upon which we might consider the question of habitability to depend, according to our suppositions of the state and wants of intellectual beings, it is as yet unknown, as much as that of the medium in which they live corresponding to our atmosphere; therefore, we may suppose [illegible] con-

[illegible]

PART IV.

OF THE FIXED STARS.

CHAPTER I.

General considerations in respect to the Fixed Stars, and methods of registering their position.

§ 139. On leaving the solar system, to introduce our readers to innumerable systems of a like nature, of which we can only discern the central bodies, or, at least, those that by their intensity or abundance of light become visible to us, we advance from distances, though large, that are still measurable in terms of the diameter of the earth's orbit as an unit, to distances immeasurable by us, and which we must call *infinite*, because we are deprived of all lineal dimension, by which to measure them; the *Diameter of the earth's orbit disappears* entirely in comparison with them; perhaps even that of Uranus; light itself, with a velocity hardly conceivable by us, must take more than six years to perform the passage from the nearest fixed star to our solar system; and these bodies themselves be much greater than the sun, to be visible to us at a distance whence the earth's orbit subtends an angle of less than a second.

Such is the case with all the stars that adorn the spectacle of the heavens, except those which we have accounted for in the preceding parts of this work. To us all means of distinguishing them fail except their apparent magnitude, which, therefore, is the ground of their classification. By this they are principally arranged in ten classes, numbered from the first to the tenth magnitude. This magnitude, however, is only the effect of their brilliancy, and does not refer to their real size, nor is it entirely the effect of a greater proximity, as far as we are able to judge. The stars of the *first magnitude*, or which we might rather call of the *greatest brilliancy*, lose their magnitude by being seen through powerful telescopes; we must therefore attribute their apparent magnitude to the intensity of their light. The twinkling, or apparent spreading of the light, we know to be due to our own atmosphere; and when the moon passes before any one of them, the angular velocity of the moon is sufficient to make the disappearance of all stars equally instantaneous.

§ 140. If we contemplate the spectacle of the stars presented after the sun has sunk below our horizon, we find every successive evening no more than a slight difference between the time of their appearing above the eastern, or disappearing below the western horizon, while all the relative positions not heretofore recorded, appear permanently fixed, and subject, besides, only to that apparent motion, which we know to be the effect of the rotation of the earth, or the day, simultaneously with which the spectacle also renews itself. Of the gradual change, arising from the successive appearance of the stars in the east after sunset, while others cease to be visible in the west, we are already apprised, as a natural effect of the gradual progress of the earth in her orbit; for we have seen that the day, reckoned under the influence of the sun, is not the same as that referred to the fixed stars, while distinguishing, as we have done, between astronomical and solar days, or time; this has already given the measure of the daily difference, which is the element that governs the

successive appearances or disappearances of the fixed stars, in our starlight nights.

These phenomena are therefore already explained. By the adoption of the revolution and rotation of the earth they became unavoidable consequences: before these were admitted, the first appearance or disappearance of a star, compared with the setting or rising of the sun, formed an important part of astronomy, and was therefore observed and recorded as such; but to repeat any part of this subject here, would be as useless as the recital of the old and exploded theories of the Ptolemaic and Tychonic systems. We may suppose our readers sufficiently acquainted with the general appearances, and with the bearing of this subject, both by common observation and from what has been already said; we, therefore, devote this part rather to the developement of the finer effects and results obtained by astronomy, and which are more peculiarly due to the latest times, and to more perfect instruments.

§ 141. In sections 11 and 12, we have stated how the positions of the heavenly bodies are registered by *Longitude* and *Latitude*, when referred *to the Ecliptic.* The permanency of the fixed stars in the same position, however, presents a mode of registering them more easy and simple, both in the determination and in practical application. The apparent motion presented by them, in consequence of the daily rotation of the earth, furnishes means to determine their relative position, by the difference of the time of their passage through a given plane, which, as we have seen, the meridian of any place presents to us with great facility; and the arc perpendicular to the equator and passing through the star, will at that moment coincide with the meridian itself; hence, the distance from any determined point in the meridian will also give the means required for the second datum, namely, the distance from the plane of the equator; such a determined point is the *Zenith* of any place, namely, the point of the heavens equally distant from the horizon on all sides, and consequently in the

meridian itself; to this point observation and calculation can refer both the star and the point of the ecliptic, where the perpendicular arc through the same will meet it; the difference between the two will evidently give the distance from the equator, as required. The difference of the time of the passage of any two stars through the meridian, will give what is called in astronomy their difference of *Right Ascension*. It is also usual to refer these determinations to the ascending node of the earth's equator with the ecliptic or 0° ♈; that is, the position of any star in right ascension is reckoned from this point. The star's distance from the plane of the equator, in the arc perpendicular to it, at the point thus determined, is called its *Declination*. Thus we have again the position of the star determined, upon the same principles as it is by longitude and latitude, relative to the plane of the ecliptic.

§ 142. But before these mathematical determinations were introduced, or, we may even say, were possible, a method of distinguishing the fixed stars was introduced, which habit, and ease of language in designating them, has preserved ever since; namely, to designate certain groups of them by the denomination of animals, memorable men, or objects, the figures of which were traced around them. These groups are called *Constellations*, they evidently extend over the whole surface of the celestial sphere; the same means which we have related, in relation to the Zodiac, in the first chapter of this work.

To make the enumeration of these constellations here, and to show the methods devised to assist the memory in recollecting, and facilitate the means of finding them, is not so closely connected with the elucidation of the *Principles* of the *System of the Universe*, as to induce us to enter into the details of this part of practical astronomy, which is designated by the name of *Astrognosy*; it may suffice to state that the best means of acquiring this knowledge is what is called the *Alignments*, which, by forming regular geome-

trical figures, of determined proportions, within those of the constellations, greatly augment the means of reference, and facilitate the recollection of the corresponding appearances.

Within the constellations themselves, the stars are designated in two different ways, by letters of different kinds, and by peculiar names; these last are Arabic, both in language and origin, and date from those times, when, the nations from which we boast our origin being plunged in ignorance, the spark of astronomical science was maintained by that people, now fallen back by the advance of other nations, who in their turn, perhaps, prepare themselves for the same fate.

CHAPTER II.

Reductions of the positions of the Fixed Stars. Nutation, Aberration, Precession, &c.

§ 143. Notwithstanding the general appearance thus presents a stability between the relative positions of the fixed stars, this again, on more close enquiry, is by no means the case; more accurate observations have discovered changes, some of which have been accounted for, and are periodic; and others only lead us as yet to suspect, that the variations to be accounted for, will be of a much more delicate nature than those we have treated of hitherto, as they may, under very small appearances, imply effects of very great magnitude; therefore, until the later times of more improved science, the appearances were the only object of inquiry, and the injurious conviction that these stars were fixed often caused their real motion to be ascribed to an error of observation or calculation. To suspect this was so much more plausible, as the observation is always mixed with the irregularities of which mention has been made in speaking of the earth as a planet; for it is evident, that the effect of the nutations of the earth's axis, the relative velocity of light, and of the earth in her orbit, the variation of the obliquity of the ecliptic, and the effect of refraction under different states of the atmosphere, must all combine in a variable result, producing temporary appearances, from which it is as difficult, as it is necessary, to disengage the observations, in order to arrive at some regurity. It was also natural at first to suppose, when the revolu-

tion of the earth around the sun was determined, that a change must result in the appearances in consequence of it; which, from the similarity of principle with the parallaxes, resulting from our position upon the surface of the earth, was called the *Annual parallax*; this, therefore, has been an object of inquiry even until lately; in fact, if we consider the sun as a fixed star, around which the earth revolves, we must reduce the appearances of other fixed stars to this centre, in the same manner as we have done in the beginning, in rendering an account of the motions of the planets around the sun, to discover the regularity of the whole system, either at rest or in motion, which they may present. But if in that case our knowledge has enabled us to start from a determined result of the science, retracing the steps of discovery, and to obtain as a final result the satisfactory explanation of the appearances, we are not yet able to follow the same manner of proceeding here; we can as yet but collect appearances, and by accounting for the influences which we are acquainted with, reduce them to that state in which the resulting system is to be compared with observations, to test its truth or fallacy, or, by the regularity of the deviations, lead to any true system that may lie hid in it.

§ 144. Our first task, therefore, is to show the principles of the *Reductions*, or *Corrections*, to be applied to the observations. We may here, as we did hitherto, neglect treating of the *Refraction*, or the deviation of the rays of light in our atmosphere, which, as a local phenomenon of the earth, may best be treated of there. Those alterations of position which the earth suffers, either as temporary perturbations, or as an ultimate progressive result of them, must become apparent in the position of the fixed stars, in a manner combined by the relative direction of the perturbations and that of the fixed star observed. We have seen above, that the variation in the position of the moon's orbit occasions what is called a *Nutation of the earth's axis* by the moon, by which this axis describes a small ellipse, the position of which is accounted

for from the position of the node of the moon; the angle, therefore, between this and any fixed star observed from the earth, forms what we have before called the *Argument* of the effect of this nutation upon the star; that is, it determines the quantity of this effect for the time; the quantity of this influence is of course again reduced to the apparent effect produced, as referred to the plane of the ecliptic or equator, and to the arcs perpendicular to them, passing through the respective stars. Hence arises the necessity of tables, either of a general nature, which would furnish the means to calculate, for any point of the heavenly sphere, the effect of the nutation for certain positions of the node, at certain distances from each other, and to which the known position of any star is to be compared; or of special tables for each star, giving this effect, in all the directions desired, for the different positions of the node of the moon; for both cases a considerable work is evidently required to give to the results all the accuracy to be desired.

In this, we have an example of a kind of tables necessary in astronomy, different from those quoted as recording the motions of the planets, namely of such as present the results of calculations of effects under certain given circumstances, which, therefore, need only to be applied to the case; of such, a number are necessary to facilitate the work of the astronomer, and by the aid of which, therefore, the science progresses more rapidly.

All that has been said here of the *Lunar nutation* evidently applies equally to the *Solar nutation*, with the difference of the argument only, which in this last is of course taken from the position of the sun, compared to that of the star.

Not to repeat here what has already been said, we refer, for the magnitude of these effects, to the place where in treating on the earth we have already stated them.

This effect of the nutation evidently cannot take place without a corresponding influence upon the obliquity of the ecliptic; this obliquity is therefore subject to an oscillation

affecting the plane of reference, when the stars are registered by their *Right Ascension* and *Declination*; but this evidently must not be attributed to the ecliptic itself, though, in reducing the places of the stars to it, account must be kept of the angle contained at the moment between the two planes.

It will be observed, that the effect of the nutation upon the position of any star at any moment, depends on the angle between the star and the direction of the axis of the earth. This, like all effects of this nature, as we have seen heretofore, it is usual to represent by its two resultants, for either the ecliptic or the equator, and the arcs perpendicular to them.

§ 145. As one of the final results of the attractions of perturbation upon the earth we have stated in its proper place, that at the present period *a diminution of the obliquity of the ecliptic* is taking place, which amounts to near 50″ in a century; and further, that the point of intersection of the two planes of the ecliptic and the equator, or the *Equinoctial points*, is affected by a retrograde motion upon the ecliptic, or what is the same thing, that the point of 0° ♈, (from which, as we have stated, both *Longitude* and the *Right Ascension* are usually counted,) moving in the ecliptic in the inverse order of the signs, occasions the return of the appearances, as counted from that point, to precede in every revolution that of the stars themselves. This effect is called the *Precession of the Equinoxes*, and requires to be taken into account in the determinations of the positions of the fixed stars; this is done in such a manner as always to state the position of the stars, at any moment, from the point 0° ♈ occupies simultaneously; this effect amounts to about 50$\frac{1}{5}$ seconds annually, and may be represented by supposing a revolution of the axis of the earth to take place around the pole of the ecliptic, the epoch of which, according to the present data, would be 25750 years. By this motion, for instance, the pole star will be, in the year 2102, nearest to the pole, or only 27′ 22″,65 from that point.

§ 146. These influences being accounted for, the observa-

tions still leave a variation from the calculated result, the epoch of which corresponds with the revolution of the earth around the sun, or the year; this effect, however, does not correspond to an annual parallax. Bradley, inquiring into this, was lead to the same discovery which Huygens had made about the same time, from the variability of the times of the eclipses of Jupiter's satellites, namely, the *gradual propagation of light.*

'The earth, in the course of her yearly revolution around the sun, must see every fixed star successively in the same situation, in relation to the sun, as any of the superior planets; hence she will be in the different parts of her orbit at different distances from the star, the extreme of this variation being evidently the diameter of the earth's orbit; now light requires, as we have seen, $16^{m.}$ $14^{s.}$ of time to pass through this diameter, during which the earth again describes a small arc in its orbit, which at a mean is $40'',5$. If, therefore, the effects of this change of appearances, produced by the gradual propagation of light, are to be accounted for, a reduction of them to the centre, or the sun, becomes necessary, which will in its maximum present the half of the above, namely $20'',25$. This reduction or effect is called the *Aberration*. To account for its direction and quantity in the different positions of the earth, we must have recourse, as in all cases of motion, to the mechanical composition of the result from the two motions of the earth and the light; these cause us to see the star in a direction indicated by the diagonal of a parallelogram, of which the two sides would be in the proportion of the two velocities, and under the angle which they form with each other at any given time; the above reduction is the maximum in the opposite directions, and can only take place when the direction of the earth's motion is perpendicular to the direction of the star; it may evidently diminish to nothing when the direction of the earth's motion and that to the star are in the same line. To reduce this appearance again to the reality, tables have been made, as useful, nay

necessary, auxiliaries, to the methods of astronomers, upon the same principles as stated for the Nutation; the arguments of these are the longitude of the sun, they therefore serve for any angle between the sun and the star, to find the magnitude of this reduction.

§ 147. It will appear immediately evident, that the planets must be all subject to the different reductions treated of in this chapter, in the same manner as the fixed stars, for we have seen that they are all dependent on the effects of the motion of the earth itself upon the appearances, and in this case the result is rendered more complex by the proper motion of the planets themselves.

§ 148. Thus every star that we observe is to be freed from these effects, in order to present, as an ultimate result, the quantity and direction of its motion. After these reductions, there still remains, in a great number of stars, certain constant but small motions, of which, as yet, no law has been discovered, that will give an explanation applicable to all the stars, or to any great number of them. Peculiarities occur in these motions, which are called the *Proper motion of the fixed stars;* the greatest amount to about 5′ in *Right ascension*, or 20″ in *Declination;* they will in future be of the greatest interest, because they evidently lead to the discovery of systems of heavenly bodies in revolution, similar to the solar system; of these, and several other phenomena and appearances of the fixed stars, it will be proper to mention a few hereafter to show their peculiarities. Recent and accurate observations have discovered a great number of them.

CHAPTER III.

Distribution of the Fixed Stars.—Distinction between them; and Changes they present.

§ 149. Although we are unable to ascertain by any direct means, whether the apparent greater or smaller brilliancy of the stars is due to their real magnitude or to their smaller or greater distance, we might, however, be led to suspect that the latter is in part the true cause. Under the supposition, which we besides see verified in all that has preceded, that every celestial body, or system, will be placed at a certain distance from the other, by allowing an approximately equal space to the different systems, of which we suppose the fixed stars to be the central bodies, or suns, we find their number also increases in an approximately similar ratio, with the decrease of their apparent magnitude. Such ought to be the case if any number of objects, nearly equally distributed in space, were viewed from any one of them.

The stars of the first magnitude are those which appear to us in brilliancy and magnitude nearest to *Mars*; they are generally reckoned to be fifteen in number, though astronomers vary, both in this number, and in the choice of some of the stars, some having added the four largest of the second magnitude of other astronomers; by all, however, twelve of them are considered as decidedly of the first magnitude.

More than sixty stars are reckoned of the second magnitude, upwards of two hundred of the third, and the number of each class increases as the apparent magnitude diminishes.

After having exhausted the usual subdivisions of magnitude, the field of discovery extends itself immensely, and presents no other limit to the increasing number than that of our optical powers. By the constant improvement of these instruments, a number of stars of considerable brilliancy are found to be composed of two and even of three, of different magnitudes, and there are some which have a periodical change of appearance or magnitude. We discover, besides the regularly defined stars, white spots of greater or smaller extent, either regularly round, or of irregular shape, which are called *Nebulæ.* A closer inquiry decomposes them into a greater or smaller, or often even an indeterminable, multitude of stars; these subjects deserve to be treated of more particularly, and though we cannot enter into the mathematical details, which are only for the professional astronomer, still, however, the collection of a few of the most striking facts must interest our readers.

§ 150. The most remarkable and most extended of all the *Nebulæ*, if we may call it so, is the *Milky Way.* By this denomination we designate a belt of whitish light, spreading itself with more or less breadth almost exactly like a great circle over the whole celestial sphere, and cutting the ecliptic near the *Solstices.* Some of the ancient philosophers, among whom was Democritus, declared this to be the effect of an innumerable multitude of fixed stars; and this, long before the invention of the telescope afforded a confirmation of the fact, by daily introducing us to the acquaintance of numberless small stars, and more distant nebulæ: as many as fifty thousand stars have been counted in it in a space of fifteen degrees.

It is a necessary consequence of the manner in which we find ourselves encircled by the milky way, that we occupy a place within this belt; we may therefore suppose our solar system, and the stars of greater magnitude, to form part of the same multitude of stars, which thus appear spread over a wide extended surface of a certain thickness. We may even venture to conjecture, that the position of our solar

system is not central within this belt, as visible to us, and that we are nearer to that part in which we see the constellation of the *Eagle;* because there the milky way appears broader, and the stars more scattered, than in the opposite direction, namely, towards the constellation of *Orion:* in respect to our elevation above the plane of the milky way, we may consider it as indicated by the angle supplementary to that subtended by the milky way. There is a singular coincidence, namely, that the largest star which we see, viz. *Sirius*, or the brightest star in the *Great Dog*, appears to us in the direction of a *Diameter* of the milky way. Desirous of finding a centre and motion wherever we see celestial bodies in any thing like a systematic order, we may easily suppose that some astronomer may have been prone to seek for such a centre in this star; and to consider the whole system, or accumulation of the stars which appear to us more brilliant, with all their successive gradations, down to the smallest of the milky way, as belonging to the same *Nebula* or cluster of stars, appearing so great, principally on account of its proximity. But to determine entirely any single point of this vast inquiry, exceeds, as yet, our means. Astronomy is too young for an epoch such as we cannot refuse to ascribe to motions of such immensity; ages much more remote, and means much superior to ours, may determine them.

§ 151. Thus we have again arrived at the limits of our actual or exact knowledge, and we are obliged to stay our judgment, until new and more accurate determinations furnish numerical data; but in the principles that have directed us in the solar system, the success obtained by the reduction to exact mathematical principles, has shown us the road to our object; here we have no more to do, than to state some of the individual data obtained, as we have hinted. Thus we find, first, *Stars*, which, under the semblance of *Fixed stars*, without any comet-like appearance, or any motion apparent from the earth, have been positively seen and determined at certain epochs, have never been seen before, and never again

appeared. Several ancient accounts of such appearances are extant, the discussion of which we may leave to the historian of astronomy; we shall therefore only quote three such, recorded by the most eminent astronomers. *Tycho Brahe* observed, in 1572, a star in the *constellation of the Cassiopeja*, which appeared suddenly, of a brilliancy about equal to Venus, and after an ephemeric existence of about eighteen months, during which its brilliancy almost constantly diminished, was lost entirely, without having changed its position at all during the time of its visibility. Suspicions of former apparitions of this star are not sufficiently supported to determine any epochs for it. From October, 1604, to October, 1605, a new star in the foot of the constellation of *Serpentarius* occasioned Kepler to give an ample discussion of his observations; its appearance was also sudden, and in great brilliancy, diminishing continually until its final disappearance. A third star of similar ephemeral appearance was seen in the head of the Swan, in 1670.

A second peculiarity of some of the fixed stars, and perhaps somewhat analogous to the preceding, is the periodic appearance or disappearance of them; of these, we know at present only six, namely, *o* in the *Whale*, *x* in the *Swan*, the changeable star in *Hydra*, one in the *Crown*, one in the *Lion*, and one in *Virgo*.

A third kind exhibits periodic changes of light without disappearing entirely. Such are *Algol* in the constellation of the *Head of Medusa*, β in the *Lyre*, η in *Antinous*, δ of *Cepheus*, α of *Hercules*, and some others; of some of which the epochs of the changes of light have been determined with a considerable degree of accuracy.

A fourth subdivision may be made of such as have formerly shown a variation of brilliancy, but which now appear invariable in their aspect, as one in the breast of the *Swan*, and one in the constellation *Virgo*.

Lastly, some appear to present a constant diminution of light, as α *Draconis*, δ *Ursæ majoris*, β *Aquilæ*; while others

appear to continue rather increasing in light, as one in *Sagittarius*, and one in *Pegasus*.

§ 152. We again see all the combinations which are possible in their variability of light, in some measure, presented to us in actual observation; we may still quote some details, with the view to give some idea of their possible causes, and even the scope of speculation with which they may supply the imagination.

The star *Algol*, quoted above, has a period for the changes of his light, that is well ascertained, of 2 days, 20 hours, 49 minutes; of these, 3½ hours are employed in diminishing its light from the second magnitude to the fourth; the same time is again occupied in its return to the second magnitude, in which, therefore, it appears the rest of the time. For such a change, we might be inclined to account, by the revolution of an opaque body of considerable proportional magnitude around the fixed star, or sun, when we consider, that were we to see our solar system from a distance, the passage of the larger planets before the sun might present an analogous phenomenon; at the same time the opaque and smaller body must disappear to us, in the proximity of the luminous central body, as we observe in the approach of the planets to the sun; and the result, at our distance, where both bodies no longer subtend any ascertainable angle, can only show itself as a diminution of the light which the fixed star itself presents.

§ 153. That such may, and we might say, must, be the case with many, if not the most of the fixed stars, in a variety of gradations, and differently from points of view other than the earth, or our solar system, we have no reason to doubt, upon the simple consideration of what our own solar system might present at some proportional distance; perhaps, however, there may be a peculiarity in our system, in the circumstance of the proportional smallness of the planets, which we must suppose to be varied in different proportions in

the universe, like all the circumstances in nature. In the constellation *Cetus*, or the *Whale*, two stars present us changes of light; the one of them passes, in about three hundred and thirty-three days, through all the changes from the second to the tenth magnitude.

The star in the constellation *Virgo* varies from the sixth magnitude down to the lowest, when it becomes invisible in common telescopes; in forty days it increases from the smallest magnitude to its greatest light; in this it remains fifty days, while it remains of its smallest magnitude fifty-seven days; so its whole epoch of change is about one hundred and forty-seven days. Upon the variable star in the *Serpent* observations have been made since 1662, by many astronomers; these give four hundred and ninety-four days for the epoch of its change of light, between the fourth magnitude and less than the eleventh. The variable star in the *Northern Crown* furnishes an epoch of three hundred and thirty-five days, varying between the sixth magnitude and nearly vanishing.

The variable star in *Hercules* has a period of fifty-nine days. That in *Sobiesky's shield* nearly sixty-nine days. The star β in the *Lyre* has an epoch of only about six days and a half; η *Antinoi* of seven days and a quarter; δ *Cepheis*, an epoch of about five days and one-third. The variable star in *Aquarius* has an epoch of three hundred and eighty-two days.

§ 154. We might enlarge the register still farther, but without particular advantage in this place. Much labour has long been bestowed upon this interesting subject. It is evident that if the changes of magnitude, light, or brilliancy, as we may choose to call it, should be attributed to a periodic approaching or receding from us, we should have to lend to them a velocity too disproportionate to all those we are acquainted with, in a direct line backwards and forwards, of which we have no example, and which has little probability; we may therefore rather suppose the suggestion made above

in respect to *Algol*, to apply in a general manner to these changes of light, namely, the revolutions of opaque planets around them, which weaken their light periodically. To explain the yet incomprehensible phenomenon of the sudden appearance, in their greatest light, of the three stars first mentioned, with our present means of philosophy, we must have recourse to the supposition of an extraordinary destructive catastrophe, of the ignition of a whole world, of a magnitude much superior to ours, or even a whole system similar to the solar system, at an infinite distance from us; for the magnitude as well as the fixity of the phenomenon, which presented no annual parallax, evidently leads to both these suppositions. Unwilling as we are to admit that nature would destroy her own most extensive and splendid works, we are, however, even on this supposition, within the circle of our experience, in the small scale of our daily observations, where successive changes in the state of matter, apparently present us with corresponding phenomena of continued destruction; and we have been led to the discovery of some of our fellow travellers in the solar system, the small planets, by the same supposition. Considering this catastrophe from the elevated position which we may assume, such a phenomenon among the multiplicity of worlds, in the long interval of time, might be considered no more extraordinary, or against the order of nature, than the death of a man, whose brilliant or moral attainments and actions appeared to deserve for him continuance in life, and rendered the benefit of his constant activity desirable to his fellow men. Future reappearances alone can lead to the discovery of epochs, if there should be any, for their appearance.

§ 155. Lead by observations and calculations to that station, where the astonishment at first excited by a new phenomenon is resolved into reflections upon its causes, based upon theories and calculations, which have represented the previously observed phenomena with an accuracy at first hardly suspected, we are not astonished to see fixed stars, or

suns, as we are accustomed to consider them, performing revolutions around each other; that is, the discovery of *satellites to the fixed stars*, as *Mayer of Manheim* first announced these phenomena, about half a century ago. Instead of opaque bodies, luminous bodies, that is, such as are of sufficient mass to become luminous by their heat, also revolve around each other, or rather, around a common centre of gravity, exactly as we have found the sun and the planets, and the satellites and their primary planets; so we must expect to find other bodies, whether luminous or opaque, revolving around suns, and varied in magnitude and number, as far as nature is varied in all her works.

Such revolutions have really been found to take place between most of the *double stars; Mayer*, in observing the double star α *Hercules*, found it to present very different appearances from those previously recorded by *Bradley*, who saw the small star preceding the principal and larger one, while he found it evidently following; it had before appeared only half a second distant from the larger, he saw it at 7″ distance. Soon after, several more such *secondary stars* were discovered around it. Mayer, on comparing the situation of the secondary star, of β *Cygni*, with that given by the observations of *Flamstead*, found it to have gone northerly 19″,3. These stars have since performed the greatest part of a revolution around their centre of gravity; as they are, besides, among those that have what is called a proper motion, it will at once appear that these two circumstances must combine together in the production of this change of place.

Thus we have by gradual steps, and almost unawares, arrived at the result or phenomenon of solar systems, perhaps even composed of more than one sun, with planets and satellites, tracing a path in the immensity of space, in a revolution as yet inextricable; and, as we must conclude from the slow path, in an epoch exceeding by far all our ideas of historical annals.

The change of place of secondary stars is also observ-

able in stars having no proper motion. The star *Arcturus*, which by common means presents a great brilliancy, or, as it is usually called, magnitude, Mayer immediately found to contain a whole cluster of such stars; these secondary stars are generally of the eighth, ninth, tenth, and even lower magnitude; of a pale and quiet light, varied in its colour. As more such examples of revolving double stars were immediately quoted, besides the above, α *Arietis*; *cor Caroli*; *Regulus*; δ *Bootes*; ε and γ *Delphini*; η *Andromedæ*; μ and γ *Cigni*; β *Lyræ*; ζ *Aquarii*; ω *Pisces*.

The same discoverer also announced stars to be single at his time, which in other catalogues were stated as double, thus recording the moment when, by their situation, their secondary stars were hidden behind the primary; such was the star ϵ of *Sagittarius*.

§ 156. These observations have since been pursued farther, and some of the data more accurately ascertained; and systems of three fixed stars have been found to revolve around each other. From the fact we have stated, that minute quantities are all that we can see of motions and bodies, undoubtedly much superior to all those we have treated of in detail, in the three preceding parts of this volume, it is not astonishing that the exact determination of the laws and phenomena has as yet escaped our investigation. But what we have discovered in the magnitudes, positions, and motions of the fixed stars, that is, in all their proportions of mass, motion, and geometric position, is so closely connected with what we have found on a small scale in the solar system, that we neither can, nor indeed would wish to, refuse to find these bodies governed by the same laws, nor forego the delight of comprehending the *mechanics*, or mode of action, *of nature* in her greatest work.

No idea of magnitude or distance must deter us, when we are informed by a simple calculation, that a celestial body, in order to be visible to us at the distance of the nearest

possible fixed stars, that is, when the orbit of the earth ceases to subtend an appreciable angle, will need to be at least two hundred and fifty times larger, or more brilliant, than the sun, (that is, it must be considerably larger than the orbit of the earth,) if its visibility was dependent on magnitude alone; hence the secondary bodies of first and second rank, as we see them in our system, may also be so much more numerous, and greater in other systems.

CHAPTER IV.

Of the Nebulæ; General Views and Systems.

§ 157. THE greatest and most extensive phenomenon that can come within our observation—the last and highest in the scale of our researches, appears to be presented to us in the *Nebulous Stars.* We see, with middling good optical powers, a number of luminous spots, apparently scattered without regularity sometimes, nay, generally, surrounding one or more stars; with increased optical powers, this light is decomposed into a number of stars, increasing with our power of vision, exactly as we have stated to be the case with the milky way, in which some of these nebulosities appear to lie. A near examination of the appearances of the constellations of the *Pleiades* and the *Hyades* has already presented us an image of this gradual increase of distinctness of vision; these groups of stars, which to some visions present a confused light, while others are capable of counting a certain number of stars in them, so that they become nearly a test of the different optical powers of different persons, are resolved into a certain determined number of stars; but such is not the case with the nebulæ, the full subdivision of which appears in most cases to lie beyond our reach, and greater optical powers continue to show always a greater number; so that some of them, after the application of our greatest powers, still have a mere whitish light; we also discover more such nebulæ with greater optical powers.

The oldest discovered nebulæ is that in the *belt of Andromeda,* which was seen in 1612, and is suspected of having

been seen before; its form is irregular; it occupies about one quarter of a degree, with three white, pale, and unequal radii. Another is 1° 10′ south of this, in the same constellation, but only about 1′ in diameter.

The most remarkable is that in the constellation of *Orion*, discovered by Huygens in 1656; its form is also irregular, somewhat approaching the shape of a hand, it presents, by its brightness, the gradual disclosure of an increasing number of stars, and an increase of the light in the other parts, which occasions the appearance of a change of form. It is one of the finest spectacles for contemplation with a great magnifying telescope; it shows seven stars distinctly, or more determinately assignable. Not far from this is another nebulæ, which, with an apparently central star, shows a whitish disk nearly circular; but it is very small in proportion to the preceding.

Præsepe in the constellation of *Cancer*, a collection of thirty-six distinct small stars, when viewed with the telescope, presents to the naked eye the appearance of a nebulæ, for want of sufficient power of distinction.

Between the constellations of the *Balance* and the *Serpent* is a nebula composed of an immense number of stars of very small magnitude, becoming more dense as they are nearer to the middle, where they are confounded in a strong whitish light; nearly similar to this, is one in the constellation of *Berenice*.

The number and variety of these assemblages of stars, that have been observed since the perfection of telescopes, have constantly increased; at present, the catalogues present upwards of one thousand, of varied forms and magnitudes. The southern hemisphere has presented a number, and among them two remarkable ones, which become visible to navigators on approaching the south of Africa, or the straits of Magellan in America. From this latter circumstance they have received the name of the *Magellanic clouds;* they present the appearance of a smaller and a larger white cloud, exactly like the milky way.

Two other places in the southern hemisphere have excited some attention among the navigators in those seas, by the appearance of particularly strong darkness, from the complete want of stars between places abounding in them, as between the two branches of the milky way, &c.

§ 158. After having obtained, by actual observations upon the double stars, the positive proof of the revolutions of suns around suns, and found our expectation of the immense variety of the combinations, which nature may display in the arrangement of the heavenly bodies confirmed in the astronomy of the fixed stars, as in our solar system, by the later discoveries made in it, we may be allowed to venture forward with boldness, in our speculations upon these multitudes of stars, which we find scattered over a space immeasurable even by imagination, and realizing the idea of Infinity. In the vicinity of our solar system, the minutiæ of which have been so interesting to us, we have found a number of others, of which the details pass unobserved by us, although we can observe some general results in the double stars, or we might even say compound stars, and assign probable causes for other appearances, as in the case of the changeable stars.—With all these, we have found ourselves in such an immensity, both of number and distance, by extending that system in which we are to the limits of the milky way, that we are prepared to consider those different assemblages of stars reduced to mere confused light, by the immensity of their distance, as assemblages of fixed stars, with their secondary suns and planets, of the same kind as our own nebula, as we may now allow ourselves to call all our brilliant scenery of a starlight night, as far as the naked eye discovers it. Thus we find *Infinity* filled with systems and collections of worlds, in such numberless quantities, that we might be tempted to suppose them to be created successively; as we may be sure that even light, to bring us the tidings of their existence and form, must travel thousands of centuries to reach us.—Only one difficulty might occur to some readers in this,

namely, the supposed necessity of the immense magnitude of the fixed stars to render them visible at such a distance; but upon this head we must observe that it is a general optical phenomenon, that the visibility of all objects whatever, depends not on their magnitude, but on the contrast of colour, or light and shade, which they present in relation to the surrounding objects.

§ 159. In reasoning upon the various subjects of astronomy, we have gradually arrived at a station where all means of distinction and judgment leave us, except direction, forms, and time; even magnitude and number becoming indiscernable. These mathematical ideas are therefore the only means of registering these, the greatest phenomena of nature, until, if possible, we should acquire optical powers capable of discovering changes that may lead to the knowledge of magnitudes. To this elevation of scenery, and abstraction, the human intellect and its laws of reasoning have led us. By the Copernican system, and the discovery of universal gravitation in our solar system, we have obtained the knowledge of our proper station in the universe; if we find ourselves occupying no very predominant place, and still see such an abundance of means, objects, and beings, in penetrating into the details of the creation on earth, what a more elevated idea must we conceive of the different and evidently superior parts of the universe; should we refuse to admit them as destined to the enjoyment of intellectual beings, when we see so many enjoyments to be derived already from what lies near us? The number and the gradations that we observe on earth, we cannot refuse to admit to the universe at large; and when we presume to consider ourselves as the *highest beings* in this creation, we do no more than any of the animals around us evidently proves that it does on his part; namely, to lay value upon every object, only in that proportion in which it has value for that animal; and his fears are, like ours, directed against superior means of any kind. The general tendency

of all creatures towards happiness is a sufficient proof that a benign aim lies in this great *Whole*, for otherwise our moral and intellectual faculties and tendencies would be treacherous to ourselves—a contradiction in morality as palpable as magnitude without extent would be in the physical world.

§ 160. In leading our reader through the extensive and varied sceneries of the Universe, from eminence to eminence, we have stopped occasionally at prominent stations, to cause him to bestow a general retrospective glance, and some reflections upon the connexions of the results with the surrounding parts—the moral world, the intellectual assistance which astronomy has lent, and the benefits it has already bestowed upon human society. Arrived, in our journey, at the very limits of our physical and intellectual means and abilities, where no guide remains to us, but geometric position and time, all simple magnitude of any kind being reduced to the mere possibility of indicating its presence by an indivisible point; the immensity of this Universe, and the multiplicity of worlds, with all their infinity of creatures and objects of enjoyment, will, I hope, have occasioned some elevation of feeling, free from the mean attachment to physical appearance and animal enjoyment.

May it be allowed, from this station, to make a digression to overthrow a prejudice that is still often thrown out against astronomers, as unreasonable as ignorant, or, perhaps, malicious, which pretends to consider them as irreligious, and astronomy itself as contrary to religion, in an age which boasts of civilization and enlightened liberal views; a proof of the slow steps with which the general mass of mankind follows the strides of the reflecting man, for whom mental occupation and enjoyment has more value than mere corporal pleasure.

Truly, there must be a difference between the religious ideas of a man habituated to consider all these worlds not as spread out for his idle gaze; whose mode of acquiring his knowledge of them is grounded upon truths as undeniable

as existence itself, and seeking in all a rational aim; while he finds immutable laws and order, he cannot refuse them to be the abodes of intellectual beings, endowed with faculties similar and superior as well as inferior to his, capable of feeling happiness and enjoying the same contemplations, and, we might say, capable of exchanging reflections with him. Such ideas are different from those of the ignorant, and mostly corporally feeling man, who, in his presumptuous arrogance, considers himself the master of the creation, and foolishly looking down upon it as made for him alone, still trembles at every flash of lightning—an accidental phenomenon of our atmosphere, too trifling to come under our consideration here; —who, at the same time that he pretends to believe himself the aim of all, experiences every instant his mean dependence upon every one of the numberless combinations and incidents resulting from the laws of nature, without, and even against, his control.

Will not a man, with the elevation of ideas which this science gives, be induced with more liberal ardour and cheerfulness to add his share in promoting the good to which he sees all this order of the universe tend? How much greater is the satisfaction of doing good from these motives, than from the selfish impulse of fear of punishment, or hope of reward? How much more cheerful is life itself, passed under these impressions, than toiled through with fear and trembling? Under such impressions, duty constitutes a pleasure, and misfortune loses its weight.—What an extensive life exchanged for the contracted one in ignorance! Must not such religious ideas be of more practical utility to humanity—more conducive to social feelings and happiness—than those grounded upon fear, and with the secret reserve of ultimately mending any wrong, ever so premeditated, by an empty *Formula*, a tardy *I believe*, without actually knowing *what*? Capable of rendering to himself account of the immutable laws which maintain this eternal order—convinced of its great and benign aim, he cannot be frightened by him, who, instead of hold-

ing out the cheering prospect of moral enjoyment in the use of the human faculties to the promotion of moral good, lending to the Deity the vilest passions of man, dares to threaten with wrath and vengeance, in the name of the great maker of all this, for every idea different from his.

§ 161. It will be proper to touch slightly upon a subject which is not yet admitted into the domain of astronomy directly, nor decided really to belong to it, namely, certain appearances of opaque bodies between us and the sun, or masses of light, apparently more remote than our atmosphere; such are, for instance, those of temporary large nebulæ, or streaks of light similar to the tail of a comet, though they may have been visible for several hours; they belong certainly to the class of atmospheric meteors, as well as the phenomenon of aerolites, of which it was also first thought, by some, that they were to be considered either as thrown to us by the moon, or as comets revolving around the earth; both suppositions which what has been said heretofore in this book will evidently prove impossible, without needing discussion; all these are evidently explicable, upon well averred principles of natural philosophy, to be chemical products of our atmosphere.

Only such phenomena as presented a longer existence, or seen from several places, either had no parallax, or only such a one as would refer them to a great distance, are worth noticing in an astronomical light, because, being accidental, they easily escape the notice of astronomers; therefore, also, the most of the accounts of objects of this nature being from spectators not sufficiently masters of scientific principles to guide their judgment, they must be considered and discussed with great diffidence.

At various times, black spots of small dimensions, similar to planets or satellites, have been observed upon the sun; thus, in 778, the 17th of March; in 807; in 1160; and in 1607, the 28th of May; a dark body appeared to pass before the sun, which was suspected to be Mercury, but could not be it according to his well known course; the latter observation

was made by Kepler with the naked eye, and therefore the body must have been much greater than Mercury; later observations of 1762, and 1764, have shown spots of one half of the sun's Diameter, passing in a few hours over the disk of the sun, by gradual and regular motion, having the appearance of round opaque bodies, but we know of no celestial body agreeing with these phenomena; the suspicion of a satellite to Venus, and of a planet between Mercury and the sun, have of course been alleged in attempting to explain these phenomena. In the year 1245, a brilliant star was seen in the south, in the constellation of *Capricornus*, of a planet-like appearance, which, in consequence of its reddish light, was taken for Mars; but this was inconsistent with the true place of that planet.

Separate accounts of these novel appearances, or stars, have even been written, in which these phenomena are recorded, together with those of the stars of temporary appearance, of which we have spoken above. Attention and an accurate register of the times and appearances, will, in the succession of time, clear up this doubtful point, and discover periods of return, if there be any regularity or permanency in them; until then, they must be considered as merely accidental as well as unexplained appearances, upon which our imagination is not yet furnished with any means for speculation, and which appear not to have any influence, or, as we might say, not any co-operation in the general action of attraction in our solar system.

was made by Kepler with the naked eye, and therefore the [illegible] Mercury; later observations [illegible] spots of one half [illegible] in a few hours over the disk of the sun, by gradual [illegible] having the appearance [illegible], but we know of no celestial body [illegible] these phenomena; the suspicion of a satellite to Venus [illegible] a planet between Mercury and the sun [illegible] in attempting to explain these phenomena. [illegible] 1215, a brilliant star was seen in the south [illegible] of Capricornus, of a planet-like [illegible] of its [illegible] light, [illegible] taken [illegible]; but this was inconsistent with the true place of that [illegible].

[illegible] these novel appearances, or stars, [illegible] in which these phenomena are recorded, [illegible] those of the stars of temporary appearance, [illegible] we have spoken above. Attention [illegible]

PART V.

OF THE EARTH.

CHAPTER I.

Appearances produced by the Revolution and Rotation of the Earth, and their consequences.

§ 162. Let us return from the infinite distances which we had reached in the preceding part, to devote a more close attention to the planet which we ourselves occupy, an intimate knowledge of which, therefore, interests us so much. In the chapter on the earth, as a planet of the solar system, the most general features of its revolution and rotation, and its connexion with the moon, have been shown.

We have observed that the motion of the observer on the surface of the earth, in consequence of its rotation around the axis, was the subject which required the greatest attention, as it has the most extensive effect in rendering the motions apparently complicated, and requires us to substitute for the fallacious judgment of our senses, the mathematical reasonings which account for a phenomenon in all its bearings.

We shall here, at first, present some farther details, the consequences of the double motion of the earth around

the sun, and of its rotation. These two motions, their direction, and the inclination to each other, under which they take place, do not appear to us as yet to have the same necessary connexion between each other, as we have found in the most of the celestial revolutions; at any rate, our object in this chapter can only be to show the effect resulting from the one or the other, or their combination.

The sun occupying one of the foci of the ellipse of the earth's orbit, if a line be drawn at right angles to the greater axis, this ellipse will be divided into two unequal parts, which, if measured by the angle in the centre of the orbit, will be respectively ninety-nine and a half, and eighty and a half degrees, and this difference increases about nineteen and two-third minutes in a century. (TABLE I., line 18 and 13.) This evidently occasions the earth to perform its revolution in the two parts in unequal times; so that it is now seven days longer in performing the part in which the apogee lies than that in which the perigee lies. A similar difference also occurs between the two solstices; this, therefore, constitutes a difference in the duration of what we call the four seasons of the year. It thus happens, that when the sun appears to us north of the equator, it is farther from the earth than when it is south of it, and consequently the spring and summer of the southern hemisphere of the earth are shorter than those of the northern. The immediate consequences of this upon local climates, come rather under the scope of physical geography.

§ 163. In the case of the earth, as in that of the other planets, the inclination of the axis of rotation to the orbit maintains the same absolute direction, and the same inclination to the plane of the orbit, at least with very small oscillations of very long periods. Thus then the earth is carried around the sun in such a manner as to present alternately the north or the south pole by preference, or both equally, towards the sun; this last must evidently be the case at the two equinoxes, that is, when the line from the centre of the sun to that of the

earth is perpendicular to the earth's axis. From this it follows immediately that the two solsticial points are those where the greatest inclination of the same two lines take place, and that the maximum of this angle must be 23° 27′ 57″, on each side of the plane of the ecliptic, that is, the same as the obliquity of the ecliptic. In our denominations of these four points, which determine, as we see from this statement, the changes of the seasons, we again adopt a language similar to that we have already so frequently employed, expressing the actual occurrence, by the appearance it presents to an observer in the northern hemisphere. Thus, the equinox corresponding to 0° ♈, where the sun appears to pass to the northern side of her apparent orbit, is called the *Vernal* equinox; the solstice following, which, therefore, takes place when the sun appears the most towards the north, is called the *Summer solstice;* the equinox of the fall is in 0° ♎, or when the sun appears to leave the northern hemisphere; the *Winter solstice*, finally, is when the sun appears at the greatest southern deviation from the equator. This is evidently, like all language, a mere matter of convention. This shows, that, generally speaking, the earth is only equally illuminated during one of its whole daily rotations, in two points of its orbit; that is, in the equinoxes; when in these points, therefore, all the parts of the earth have the day and the night of equal duration. In proportion as the sun appears to approach towards the one or the other pole, that is, when, by the position of the earth in its orbit, that pole is inclined towards the sun, the days increase towards the same pole, and a proportional part of the earth around that pole never loses sight of the sun; while on the opposite hemisphere, exactly the reverse takes place, the night becoming longer; that is to say, the sun is visible a less space of time, and around the pole of this hemisphere the sun remains then invisible during the whole revolution of the earth.

§ 164. Wherever we observe the effect of the sun, we have always found heat and light united; hence the circumstances first described, as producing the different length of

of all creatures towards happiness is a sufficient proof that a benign aim lies in this great *Whole*, for otherwise our moral and intellectual faculties and tendencies would be treacherous to ourselves—a contradiction in morality as palpable as magnitude without extent would be in the physical world.

§ 160. In leading our reader through the extensive and varied sceneries of the Universe, from eminence to eminence, we have stopped occasionally at prominent stations, to cause him to bestow a general retrospective glance, and some reflections upon the connexions of the results with the surrounding parts—the moral world, the intellectual assistance which astronomy has lent, and the benefits it has already bestowed upon human society. Arrived, in our journey, at the very limits of our physical and intellectual means and abilities, where no guide remains to us, but geometric position and time, all simple magnitude of any kind being reduced to the mere possibility of indicating its presence by an indivisible point; the immensity of this Universe, and the multiplicity of worlds, with all their infinity of creatures and objects of enjoyment, will, I hope, have occasioned some elevation of feeling, free from the mean attachment to physical appearance and animal enjoyment.

May it be allowed, from this station, to make a digression to overthrow a prejudice that is still often thrown out against astronomers, as unreasonable as ignorant, or, perhaps, malicious, which pretends to consider them as irreligious, and astronomy itself as contrary to religion, in an age which boasts of civilization and enlightened liberal views; a proof of the slow steps with which the general mass of mankind follows the strides of the reflecting man, for whom mental occupation and enjoyment has more value than mere corporal pleasure.

Truly, there must be a difference between the religious ideas of a man habituated to consider all these worlds not as spread out for his idle gaze; whose mode of acquiring his knowledge of them is grounded upon truths as undeniable

in the straight line joining the centres of the sun and earth; that is, they will have the sun in their *Zenith:* This will be the case until to those points that have the same latitude as the Solstices, where this phenomenon can evidently occur but once.

As no more than the half of the surface of the earth can be illuminated at once by the sun, it is evident that in the neighbourhood of each of the poles there must be always a part which loses the sight of the sun; its distance from the pole will correspond to the deviation of the sun from the equator; the part of the earth subject to this phenomenon must extend, on both hemispheres, to such a distance from each pole, as will correspond to the obliquity of the ecliptic, or 23° 28′ nearly. The part of the globe around the pole, thus marked off, is called the *Frigid Zone*, which evidently occupies the same distance around both poles.

All the intermediate parts of the two hemispheres of the earth, namely, from the distance 23° 28′ (approximately) from the equator, and from the poles on each side, where all the changes between the two extreme cases stated take place, form the *Temperate Zones* of the earth, which, like all that we see about us of a medium quality, form the greatest portion of the earth's surface.

The principal distinctions which have furnished these divisions are strikingly evident, and their limits are well determined. They form less circles of the terrestrial globe, which are parallel to the equator, whence they have, like all such circles of the sphere, the general denomination of *Parallels*. The two parallels dividing the torrid zone from the two temperate zones are called the *Tropics;* those surrounding the poles, at the same distance from them as the tropics are from the equator, are called *Polar Circles*, which, therefore form the division between the temperate zones and the frigid zones. These denominations are taken from the phenomenon that most affects the feelings and the enjoyments of man, namely *Heat*, which is the principal cause of vegetation

on the earth, as well as of the existence of the animated creation. The gradual succession of changes resulting from it, and acting upon human beings in the different parts of the earth, are of great importance in determining the manner of their existence, and, therefore, in fixing their moral or intellectual state; these, however, are always more equalized by the progresses of civilization. A more close investigation of this subject is of great interest in the history of civilization, as well as in the future improvement of the human race; but is entirely foreign to the object of this work.

If the whole surface of the earth be supposed to be represented by the number 1000, the proportion of the superficial contents of the different Zones will be as follows:

The *Torrid Zone* will be expressed by	398
The two *Temperate Zones*, ...	520
The two *Frigid Zones*, ...	82

Or more accurately: According to the most modern determinations, the Torrid Zone is equal to 202784

Temperate	264364
Frigid	42118
Total,	509266

Though no well-ascertained and regular law for the diminution of the temperature of climates has as yet been discovered, an approximate result appears to be presented by the product of about 49° of Farenheit's scale into the square of the cosine of the latitude, (that is, the square of the distance from the earth's centre to the perpendicular drawn from the point on the earth's surface to the plane of the equator).—But this variation is of course very much influenced by local circumstances, which act according to the general laws of nature, upon the absorption and communication of heat in different substances, and their different state.

The greatest and most general influence of a local nature is the elevation of the place above the level of the sea, which is to be considered as the general surface of the earth, ac-

cording to well founded philosophical principles. For equal latitudes it is always found that the heat diminishes with the increase of elevation above the sea; and even in penetrating into the earth, we find an augmentation of the mean temperature.

§ 166. The variation in the length of the day and night, increases constantly from the equator towards the poles. Under the equator, the sun always appears to rise perpendicularly from beneath the horizon; the days and nights will thence be equal all the year, with that difference only, which we have in its proper place designated by the name of the variation of the equation of time; only the half of which is of course sensible in that difference.

When the point in the equator, which is under consideration, is in the plane passing through the centres of the sun and the earth, and perpendicular to the equator, or in its transit through the meridian of a place, the zenith distance of the sun will always be equal to its angular distance from the equator; that is, to its declination. As we proceed from the equator towards either of the poles, the length of the days becomes unequal, and their difference increases with the latitude. In every place between the tropics, the sun will pass twice every year through the zenith; it will, therefore, be seen on both sides of the zenith alternately.

At the parallels of the tropics, the limits of the greatest declination of the sun, or about 23° 28′ on either side of the equator, the sun reaching the latitude of the place only once in a year, it will appear once every year in the zenith, and all the rest of the year on the same side of it; its greatest distance from it can be no more than twice the obliquity of the ecliptic. The first of these cases takes place at the solstice of the same name with the latitude of the place; the last at the corresponding solstice in the opposite hemisphere; that is, in the northern hemisphere, the first at the solstice of *Cancer*, and the second in the solstice of *Capricorn;* and at these points the difference between day and night is already

nearly two hours from the equality, or 12 hours. These same points will also determine the times of the longest or shortest day, for the whole hemispheres in the same order; the difference increasing constantly with the latitude, until at the polar circles the length of the day or the night will vary the full time of the earth's rotation around its axis, in the above extreme cases.

Besides this, at the two polar circles, the sun appears nearly two days above the horizon by the mere effect of the refraction of our atmosphere, of which we shall have to speak hereafter; and other phenomena peculiar to the atmosphere of these climates vary the appearance of their days and nights differently from any thing we are accustomed to see in the middle latitudes.

§ 167. We have often been obliged to make use of certain denominations of planes and lines, transferring pure mathematical ideas to their application in the material world. These are so generally known, that I considered it permitted to make use of them before I had accurately defined them, in the manner that Astronomy and the mathematical consideration of the earth makes use of them. By defining them now, we shall unite what we have said of the manner of determining the positions of the fixed stars, and the celestial bodies in general, with the geometry of the earth itself.

The plane which appears to us extending around us at an undetermined distance, and thence appears limited by a great circle surrounding every observer, is called a *horizontal* plane, and the great circle terminating it is the *Horizon*. To this plane we erect a perpendicular through the point of observation, which we consider also as extended infinitely, both above and below this horizon. The point over our head, to which we refer this perpendicular, is called the *Zenith* or *Vertex;* and that in the opposite direction, which, therefore, it is always impossible for us to see, is called the *Nadir*. Thus we form a mathematical sphere around every

observer, or point on the earth's surface, which naturally furnishes the fundamental points of reference for any position of the celestial bodies, as well as for that of different parts of the earth, so far as they mutually lead to conclusions by means of observations. Perpendicular to the plane of the horizon, we imagine a plane through the vertical line and any point whatever that may come under consideration, which we call, therefore, a *vertical* plane; and the circle to which it is referred, at any infinite or unlimited distance, is, therefore, a *Vertical circle.* The extension of any such plane on the other side of the horizontal plane, that is to say, in the hemisphere beneath the horizon, and ever invisible to us, evidently passes through the *Nadir.*

The sphere thus determined in a mere imaginary manner for every observer, obtains its determined position, and what we would consider as *Fixity*, if we were not already informed of the complicated motions of the earth, on account of the line of the *Zenith* and *Nadir* being in coincidence with the direction of gravity at every point; the horizon becomes a plane tangent to the surface of the earth at that same point. The *Meridian* of a place will form one of the vertical planes just described; we thus obtain the plane of the meridian as a second determined plane, to which to refer all the motions of the celestial bodies; its position in respect to the horizon is therefore constant, for it will pass through the zenith, and those points to which, as we have seen, the prolongation of the earth's axis is referred, namely the *Poles.* Here we join the mathematical divisions, referring to the earth, to the phenomena which we have found in Part III. Chapter 3, to result from its revolution and rotation; and we easily see the coincidence of the situation of the place upon the earth in respect to the sun, with the passage of the sun through the meridian of the place; the former being what we might call the external, the latter the internal aspect of the same phenomenon. The lesser circles of the earth, parallel to the equator, passing through any

point of its surface, and which we have seen to be called *Parallels of Altitude*, must of course refer themselves to the zenith, as above stated; for they must equally refer to the direction of gravity, cut the meridian perpendicularly in the zenith, and be tangent to the great circle perpendicular to it.

This would be the state of things if the earth were a perfect sphere, which we know is impossible, according to the laws of gravity and the nature of matter in general. The deviation of it will form an object in the next chapter, where a short general idea of the actual figure of the earth shall be given.

§ 168. The principles of the preceding section evidently give the fundamental ideas of all instruments intended for astronomical observations. We have given, by nature, for mathematical use, two circles perpendicular to each other; and their direction, together with the perpendicular to one of them, namely the direction of the gravity; if we place, therefore, by this means, the one circle in the horizon, and, turning upon the direction of gravity as an axis, bring the other, or vertical circle, into the plane of the meridian, we shall be enabled to observe *the time* when this mechanical meridional circle coincides with the direction towards any star in the course of its rotation with the earth. This observation is called the *Transit of a star over the Meridian*, which, therefore, is *indicated by the Time*. By the angular distance from the zenith at which this star will appear to pass it, we obtain what is called in astronomy its *Meridional Zenith distance*. It is evident that we may equally refer this angular distance to that from the horizon; because the zenith being distant 90° from every point of the horizon, this distance from the horizon, which is called the *Altitude*, must be the difference between 90° and the zenith distance of the same point. In thus placing our two circles, we have chosen the simplest possible case for observation, and at the same time the most efficient and most accurate. Placing the vertical circle under any required angle with this meridional position, we measure

of course, on the horizontal circle, the angular distance of any other vertical circle from it; this angle is called the *Azimuth* of any vertical, or point in that vertical. Here again we may, therefore, determine the *Zenith distance* or *Altitude* of a point in any vertical that lies out of the meridian; and this will be the most accurate means of determining it; the latter will therefore be resorted to whenever the first is not possible.

§ 169. Observatories upon the best modern construction are therefore principally furnished upon the first of the above principles; and, as simplicity always tends to accuracy, the general instrument which we have supposed above, is still divided into two more simple ones; namely: first, a simple *Telescope*, revolving with the greatest accuracy in the vertical plane of the meridian, and which, therefore, will serve to observe the time of the transit of any celestial body through that plane; this time, compared with that of the passage of any other star, will give what in astronomy is called, their *difference of Right Ascension*. These, we have seen, are all referred to the point of 0° ♈. The second instrument is a *Circle*, placed in the same plane of the meridian, and therefore vertical, or perpendicular to the horizon. By this circle the point in the meridian to which any celestial body is referred in its passage is determined; and thereby its angular distance from either the Zenith or the Horizon, and is called, as the case may be, the *Zenith Distance*, or the *Altitude*. With this circle, therefore, we determine the difference of altitude, or zenith distance of two celestial bodies; as we have seen that the poles of the earth also lie in this vertical circle, we may easily see that we thus obtain a means to refer these observations to a comparison with the place of either of the poles in this circle, or to determine their polar distance. We have here again a case similar to what occurs for the horizon and the vertex; referring these angular distances to the plane of the equator, to which we know the poles to be perpendicular,

we obtain (as a complement of the polar distances) the angular distances from the equator, which are called in astronomy the *Declination* of the heavenly bodies.

In the meridian, therefore, the coincidence of the two spheres which are formed, the one upon the horizon and the other upon the equator, takes place, and in this all the most accurate astronomical determinations are made with the greatest facility, and most near to what we may call the point of junction between pure geometry and the most invariable law of nature, *Gravity*.

§ 170. For the determination of the place of a heavenly body, when out of the plane of the meridian, we evidently see that we are reduced to our two first circles, *parallel* and *perpendicular to the horizon;* from these we must determine every other position, by the assistance of the *Time;* or by a comparison with a known point in the celestial sphere; or by means of the *Azimuthal angle* itself, by referring it to that which it would occupy in the meridian; and thence we obtain the corresponding right ascension and declination, by which it is transferred to the celestial sphere in general.

Recent improvements in optics have given birth to another kind of instrument, by which angles are measured in the plane perpendicular to two mirrors, namely, reflecting instruments; circles, or sectors of the same, are made upon this principle. The accuracy of the principles employed, and the great aptitude of those instruments to every kind of observation at sea, has brought them to so high a degree of perfection as to vie in point of accuracy with instruments of other kinds of much greater magnitude. It is evident that in this case, the angle measured being necessarily always in the plane passing through the observer and the two points observed, it will necessarily also obtain the altitudes, or zenith distances of the two points observed, and thence the time of the observations. From these, by means of mathematical calculations, we must ultimately obtain the position which any celestial body occupies, whether we may have ob-

served it in the meridian itself, or out of the same; both of these are equally possible with this kind of instrument.

§ 171. We require instruments for measuring time with more minuteness than can be done by a single revolution of the earth, not only for the general consideration of astronomy, but in the detail of observations.

To do this with great accuracy required, of course, a very advanced state of improvement in mechanical skill, and the application of natural philosophy; to this, and to the great value which the accurate determination of the time at sea possesses in the determination of the longitude, we are indebted for a very great perfection in time-pieces, both in the form of *Clocks*, as used in fixed observatories, and of *Chronometers*, which are transportable with more or less facility, according to the use they are intended for.

In treating of the subject of instruments of astronomy, we can do no more than give the very first elementary principles. These, when accuracy is desired, must be approached as near as possible; and, the greater the accuracy that is required, the nearer must the instrument used in the observations be brought to the simplest elements; mathematical science is sufficiently improved to deduce from observations thus obtained any desired result with complete accuracy, and this can never be obtained by making the instrument itself perform the calculation.

§ 172. We have seen that the meridian of any place passes through the vertex of that place, and the poles of the earth; we have also seen that by measuring the meridional altitude of any celestial body, when we know the position of the pole in the meridian we can determine the polar distance or the declination of that celestial body. Inverting this problem, and having the declination of a star well determined, we may evidently determine in return, the zenith distance of the pole, or its altitude; or the angular distance of the equator from the horizon, or of the *Zenith from the Equator*. This last corresponds to the angular distance of the place of observation

from the equator, which is called the *Latitude of the place*; for the vertical line, or direction of gravity, corresponds, as we have seen, to a radius of a sphere tangent to the place of the horizon, which would be at the same time that of the earth, if it were a perfect sphere.

It thence results, that by observing the meridional altitude of any celestial body, whose declination at the moment is exactly known, we are able to find the *Latitude* of the place. The elevation of the pole towards which this latitude counts, above the horizon of the place, is equal to it, because, on leaving the equator to proceed towards either pole, it is evident that this pole appears to elevate itself more and more above the horizon, and being a fixed point in relation to the rotation of the earth, it never passes beneath the horizon of the place in the course of this rotation. Thence the altitude of the pole is equal to the latitude of the place.

§ 173. We have already seen how the difference of *Longitude* of two places on earth is determined, by the simultaneous observation of a certain instantaneous phenomenon; one in which the moon is concerned being usually taken; because that satellite has the most rapid apparent motion; we may here quote such other methods as refer to time simply, or to transits over the meridian. If we suppose an accurate time-piece, that shows the time of one place, to be transported to any other, we shall find it to show as much difference between the meridian transits of a celestial body (for instance the sun) at the two places, as corresponds to the difference between the meridian of the two places, and considering the three hundred and sixty degrees of the circumference of the earth, to perform a rotation in twenty-four hours, we can, by this proportion, easily reduce this difference of time to the difference of the degrees of the equator, which denote the difference of longitude. It may even be indicated in either way indiscriminately.

The great velocity of the moon's motions allows us to compare the time of her transits through two meridians with

those of stars near her, in order to determine their difference of longitude; because this velocity of motion will, for the difference of absolute time of the two observations, indicate a proportional motion of the moon, for which the perfection of the lunar tables at present indicate a certain time necessary to perform it, which will again be the difference of longitude between the two places.

There is no determined point given by nature, from which the *Longitudes* on earth should naturally begin. Astronomers count from the most convenient, well determined, observatory, for which Paris and Greenwich are the most used, because the most extensive tables of those differences are given for them; for geographical maps, and particularly those on a globular projection, the meridian of the island of Ferro was long, and is generally yet, called the first meridian; and, to give it a fixed determination comparable with the observatories, it is assumed to be twenty degrees west of the observatory of Paris.

CHAPTER II.

Figure of the Earth, and its consequences.

§ 174. The general principles of natural philosophy indicate, that the figure of any body whatever must be the result of the forces acting upon the different parts of matter which compose it. For the celestial bodies, we have the most *perfect freedom of matter*, in kind, quantity, and all its general properties; we find it, besides, under the most simple and general law, namely, that of *Universal Gravitation*. Adding to this the influence of the rotation of the earth, as the datum, or individual case, under which the general law acts, in the production of the figure of the planets, we must expect to find this figure, what is called in mechanics, a *Resultant* of the combination of two compound forces; namely: the *Attraction* towards the centre of gravity of the body, and the *Centrifugal force* produced by the velocity of its rotation, on the one side; and the *Density* and *Cohesion* of the matter composing this celestial body, on the other. The equilibrium between these must produce the actual form of the celestial body.

It is evident from this, that the data to be obtained for the determination of the minute consequences, would also require a detailed knowledge of the component parts, or the kind of matter composing the celestial body under consideration; to this the two forces just stated, and well determined by the laws of celestial mechanics, would then be applied. As long as we suppose matter uniform in state, and therefore in density

and cohesion, under the influence of gravitation and the centrifugal force, we obtain a solution of the problem in a simple approximate form, as astronomy presents the planets to us, namely *elliptic* bodies. But when we consider the earth with the interest so natural to man, and even so necessary, for an infinity of cases, and minute scientific inquiries, the result in this general form has proved insufficient. Actual measurements upon the earth's surface, and observations of the mechanical effects of the earth's attraction, have shown differences, which, though they would be unimportant for celestial bodies, seen at a distance, are observable, and worth our inquiry, when our own abode, the earth, is under consideration.

§ 175. Willing as we should be to adopt, as most simple, and therefore most conformable to the laws of nature, that the axis of the earth's rotation should be exactly the smaller axis of the ellipsoid, corresponding to the effect of the centrifugal force, and due to the velocity of the rotation of a body, or planet, of equal density in its whole mass; we know already too much of the observable influence of the ellipsoidic figure of the earth, compared with that of the mere spheric form, in the modification of the general result of mutual attraction, not to be apprised of the necessity of attending to the more minute circumstances; the inequalities of the surface, and the different specific gravity of the different parts, for instance, deep sea and high mountains, may show, by the simplest reflection, that when we enter into this inquiry with means of minute accuracy, their effect must become sensible to us; and what we expected to present perfect *Geometric* regularity, must be the figure corresponding to the *Mechanical equilibrium* of all the parts of the earth.

Such is the present state of the theoretical question, in respect to the figure of the earth, to discuss which in full is the scope of mathematical analysis; we have, therefore, noticed it here as far as is possible, in laying down the above principles. It rather belongs to our plan to state some of the principal results hitherto obtained; and these, even with-

out entering into the more minute details which belong to *Geodesy* and *Experimental Philosophy.*

§ 176. The simplest idea of all must naturally be to determine, by *actual measurement*, the *Magnitude* and the *Figure* of the earth simultaneously, upon the hypothesis, that the curve generating the solid, and which, therefore, forms its principal section, is regular. The measurement of certain parts of this generating curve, and the determination of their position in it, and thence the corresponding segments which they represent, that is, the angle they subtend at the centre of it, will determine the whole curve, and consequently both its magnitude and figure; by these means the magnitude and figure of the earth can be found. Extensive works have been made upon these principles at different times, and in different parts of the earth; at first only in a rough manner, and under the supposition of the earth being a *Sphere*; but of late with means of accuracy sufficient to prove *Irregularities*, of which the causes may lie in the above principles, though we are as yet unable to bring them under a law sufficiently conclusive to decide upon the minute details of the problem. We have already stated that the best combination of the results of the various determinations of the oblateness of the earth give it in round numbers $\frac{1}{310}$ of the equatorial diameter, or according to other calculations $\frac{1}{308,5}$. The magnitudes corresponding with these same calculations and results, give to the equatorial radius of the earth 20925700 feet English, and to the polar radius 20858198; that is, for the difference between them, or the flattening, 67502 feet; the degrees on the equator become thereby 365223 feet. The two first data already suffice to determine all the dimensions of the earth.

§ 177. Having mentioned the principal dimensions of the earth, we might extend them into details, which, however, present neither special interest, nor any difficulty to the inquirer, and are omitted here: we will quote only a comparison of them with the effect of the rotation of the earth,

which passes unnoticed by us, notwithstanding its magnitude, and its difference in the principal parallels; this consists in the velocity of motion which each point in these parallels has per second, and which are as follows:—

Under the Equator, the velocity per second is 1519,4 feet.
Under the Tropics, - - - - - - - - 1014,
At the Polar circles, - - - - - - - - 605,1

Such is the rapidity of our constant motion by the rotation of the earth, to which we are entirely insensible.

§ 178. The great inconvenience felt, in the extension of the communications of the human society, and particularly in commerce, has long since pressed upon man the desire of uniform weights and measures, founded upon nature, and following a single connected system. But the physical propensity to continue in the state and habit that has become familiar, which, in the moral world, exactly corresponds to *Inertia* in mechanics, has hitherto frustrated the different attempts made at various times; and it seems that also in this particular, our approaches to reason are only by an oscillating motion, gradually coming to the true principle. A great work, executed under circumstances unfavourable to science, because it was considered with the partial eye of political ambition and strife, has deduced from the data which have been quoted before an universal unit of length measure, from which all other measures, and also the unit of weight, were deduced. From the measurement of 12° of the ellipse of a meridian, compared with other measurements in other parts of the globe, the length of the whole quadrant, or fourth part of the earth's meridian, was deduced, and the ten millionth part of it taken as a fundamental unit, (which contains 39,381022708 inches English,) and was called *Metre*. This has already been adopted extensively in the scientific world; because, besides its goodness of principle, and the great scientific operation, impartially discussed by scientific men of many nations, upon which the result has been grounded, its actual unit has been executed

with great accuracy, and in considerable number. The accuracy obtainable by this means, for such an unit of length as comes within convenient use in common life, being probably superior to that obtainable by the length of the pendulum, or other similar methods, under the supposition, that the unit being lost, it became necessary to replace it from nature itself, it is probable that the nations of the earth will agree, at some future time, at least by the medium of the men of science, in the use of this or a similar unit of measure, grounded upon the dimensions of our globe.

§ 179. By the law of universal gravitation, heavy bodies will, on every planet, and therefore on the earth, fall with a determined velocity, which is indicated in Table II. But informed as we are of the principle, that this attraction, and this fall, which is a consequence of it, depend on the distance from the centre of attraction, for which we may as yet accept in a general way the centre of the earth, and that upon the earth we find ourselves at different distances from this centre, according to the principles of its ellipsoidic form, we must be aware that the velocity of this fall must be different in *different latitudes*, and at *different elevations* above the surface of the earth. Still, to observe this effect under that form, would exceed our means of nicety of distinction.

By the *Pendulum* we have a mean to bring this force into activity, and under observation, in a form and manner which is at once easy, and susceptible of considerable accuracy. Thence we see immediately that, following the same method of inversion that must rule in all experiments of natural philosophy, we may by the pendulum ascertain the figure of the earth; though here, as in all cases of a similar nature, the many influencing accessaries must necessarily be taken into account, in deducing any result whatever.

The Pendulum, it is well known, presents a heavy body suspended from a fixed point, perpendicularly beneath which it would therefore rest by the action of gravity; but when made to oscillate on both sides of this line, it describes parts

of an arc of a circle, the radius of which is the distance between the point of suspension and the centre of gravity of the oscillating body. Its action, and therefore the time of one of its oscillations, depends both on its length and the attracting force at the point. The regularity which this time of oscillation presents, under equal circumstances, has rendered it the fittest regulator of our clocks for measuring the time; and the time of one second has been chosen, for that purpose, in astronomical clocks, to be represented by the motion of the pendulum in going from the one extreme point of its oscillation to the other, on account of its convenient length. Astronomers have made use of the simple pendulum for the measure, or rather subdivision of time, previous to its application to clocks by one of them, Huygens. Thus the benefit of our most accurate means of measuring time is also derived from astronomy.

§ 180. The different number of oscillations described in one rotation of the earth, or a day, that is, 24 hours, either *astronomical* or *mean solar* was first observed on the occasion of a literary expedition to the neighbourhood of the equator, in the southern continent of America, and often repeated afterwards. The pendulum vibrating seconds, as just mentioned, at no more than thirteen minutes of latitude from the equator, was found to be (438,69 French lines, or) 38,96 inches English; and observations made in the latitude of 79° 50′, gave for this length (441,37 French lines, or) 89,199 inches English.

Theory gives for the change of the length of the pendulum vibrating seconds, the ratio of the squares of the line drawn perpendicular from the point of observation to the plane of the equator. The difference to be minutely subdivided for each particular point of observation, according to its latitude, being very minute, no more than about half an inch from the equator to the pole, it is evident that these observations require considerable care and attention, as well as calculation, upon all the influencing circumstances; for here, as in all na-

ture, and in all circumstances which we have seen in celestial bodies, any phenomenon whatsoever is the result of a *Combination* of circumstances and effects, and the principles upon which we may ascertain the simple phenomenon, which is to be inquired into, must be, either to neutralize the others by the arrangement of the experiment, or the methods of observing it, or to keep account of these accessary influences, ascertained, and numerically determined, from theory or observation, obtained by other means.

It is evident that the quantity of the effect produced by the change of gravity between the equator and the pole, is more sensible to observation, in the manner which has led to its discovery; hence a pendulum of a constant length, will perform a different number of oscillations under different latitudes and at different elevations above the sea, and the ease which clocks afford us to count the number of these oscillations in a certain time, for instance a single day, furnishes, with great accuracy, a means to determine this other element of the calculations, by which the problem in view can again be solved,

The theory in fact shows that the number of oscillations in equal tiems is inversely as the square root of the length of the pendulum. Or to conclude the proportional gravity directly, which leads to the ellipticity of the earth, we have the principle: that the squares of the number of oscillations are as the gravity, at the different places, that is, inversely as the radii of the ellipsoidic earth.

This subject has been pursued of late with considerable ardour; observations of the pendulum have been made in a great number of places, and the figure of the earth thence deduced, compared with those of the actual measurement, made also in a great number of parts of the earth; it has been found generally corresponding to an ellipticity of $\frac{1}{308}$; they can evidently give no more than this proportion in a general form, the accurate lineal dimensions can only be obtained by actual measurement. The discrepancy between the results

from these two sources being just now the subject of assiduous scientific researches, it is proper as yet to delay all judgment upon their results and proportional accuracy.

It must of course be expected that all the inequalities of the figure of the earth must influence the results of the pendulum; and, besides, the local density will come into view, which may be as variable as the figure.

In both the actual measurements and the pendulum observations, the results are always reduced to the surface of the sea, which is considered as furnishing the surface, to be adopted as the surface of the earth in general, as has been already observed.

§ 181. In speaking of the moon, the connexion between the figure of the earth and the perturbations resulting from the mutual influences of the earth and moon has been shown; from these, we again obtain a means to determine the ellipticity itself by a return of the theoretical calculations from the observed perturbations; and this means is brought, by the present state of the science, to such a perfection, as to come into comparison with the two more direct ones which have just been treated of; it gives for this ellipticity $\frac{1}{305}$, little different from the preceding determinations. So that an astronomer, without going out of his observatory, can solve a problem otherwise requiring extensive voyages to different parts of the globe.

In a similar manner, the fall of heavy bodies upon the earth's surface, can be deduced from the revolution of the moon around it, to the coincidence of about $\frac{1}{100}$ part of a foot with actual observations; which corresponds to a difference in the moon's parallax of three-fourths of a second; both quantities within the limits of the accuracy of this kind of observations. The length of the pendulum compared to the moon's parallax gives approximate results upon the magnitude of the earth, and this parallax is determinable by observations at different altitudes; thus the astronomer can, by comparing his observations with the theory, ascertain approxi-

mately even the magnitude of the earth, and its distance from the sun and moon, without leaving his observatory.

§ 182. As we have already stated, the direction of gravity or the vertical line is a perpendicular to the horizon of the given place, and this plane is a tangent to the surface of the earth; in consequence of the ellipsoidal figure of the earth, the perpendiculars or vertical lines can therefore pass through the centre of the earth only in those two places in which the radius of the earth is also perpendicular to the surface, and therefore coincides with it; this takes place under the equator, and at the poles; in all intermediate latitudes, the radius of the earth, corresponding to the parallel, will form an angle with the above vertical line, which, we may conclude from the circumstance just stated, will be greatest at the parallel of the earth equally distant from the equator and the poles, where it amounts to 11′ 6″,4, for the ellipticity of $\frac{1}{310}$. If therefore any latitude is to be determined, as referred to the centre of the earth, it is still necessary to correct the results of the latitude deduced from altitudes or zenith distances of celestial bodies, of which we have spoken above, for the angle of the vertical line with the radius corresponding to the place; by this is obtained the real *Geocentric latitude* of the place. In all cases, therefore, where the actual position of a point upon the surface in relation to the centre is to be determined, this geocentric latitude is to be employed; such is the case in all determinations where the moon is concerned, especially in solar eclipses, occultations of fixed stars, lunar distances, and, generally, wherever the effect of the deviation of the vertical line from the radius measured at the centre of the earth may become observable.

This latitude is therefore, in fact, the same with that which is supposed to belong to a point on earth, under the most common acceptation, of latitude being the angle at the centre of the earth, between the radius of the point and the plane of the equator, when the earth is considered merely as a sphere.

The immediate results of observations of celestial bodies, arising from their reference to the vertical line, which is that from which in fact all observations must set out, is called by astronomers *Observed latitude;* this is more convenient to use in common determinations at sea, and sufficient for their aim; it is, however, necessary to distinguish between the two, in all calculations where the figure of the earth is concerned, and which may require accuracy; in common life, this distinction shares the same fate as the difference between *mean* and *true solar* time, that is, to be disregarded.

These two latitudes, therefore, progress according to different laws, from the equator to the poles, and neither of them represents for equal degrees of its angle equal distances in the meridian upon the earth's surface; thus, for instance, the point in the meridian which is equally distant from the equator and pole, would be that, where the *uncorrected latitude*, or the mere result of the observations, referred to the vertical of the place, would give 45° 5′ 33″,2, and not 45[d], (the half right angle) in either of the two latitudes.

Thus we again find that the common language represents an idea needing peculiar distinction, determination, and closer investigation, as we have found to be the case every where in the revolutions of the heavenly bodies.

§. 183. The parallax for which we have, in the preceding parts, considered the radius of the earth as invariable, is evidently also variable, from the same cause; and in the same ratio as the radius of the earth's ellipsoid; this becomes more sensible in the *Parallax of the moon*, which we have found varying considerably in consequence of its different distances in the different parts of her orbit. A different radius must subtend a different angle at the same distance, and the parallax must increase or decrease in the same ratio as the radius. At the equator, therefore, this parallax is the largest, and is therefore that which is given in tables and nautical ephemeries, from which, therefore, an observer in any other latitude has to deduce the parallax corresponding to his place; and this is, of course,

further under the influence of those changes which correspond to the inclination of the radius of the earth towards the line of vision; that is, the horizontal parallax corresponding to the latitude, will again change with the angle of elevation, or the zenith distance of the moon; and from this corrected result the effect in any other plane, as for instance in right ascension and declination, latitude and longitude referred to the ecliptic, or in the inclined plane through the moon and sun or a fixed star, is to be deduced; the former of these are necessary in solar eclipses and occultations of fixed stars; the latter in lunar distances.

The smallness of this change in the radius of the earth, when compared with the distance of the *Sun*, renders its effect almost entirely insensible in observations upon that body, and only very minute calculations, which still exceed the accuracy obtainable in common observations, take it into account in that case, and in those of the nearest planets.—Here then we have the last step which the astronomer is obliged to make, to calculate with minute accuracy, from mathematical theories, applied to the motions of the celestial bodies and their appearances, that spectacle which they must present to him at any particular point on earth; he has still one difficulty to encounter similar to what he meets daily in the moral world, namely, the discussion of the effect of the medium through which this spectacle is presented to him, that is to say, the *Atmospheric Refraction*, of which we shall state the principles in the next chapter.

§ 184. We should still have to discuss a subject of considerable importance, if we had sufficient data to come to a conclusion upon it, namely, the actual *Density* of the earth, compared with either one of its component parts that comes under our immediate cognizance, as it is habitual to compare them among each other by means of the density of pure water.

In Table II. line 15, the density of all the planets has been given, compared with that same unit, by means of the result

of the measurement of the solidity of the mountain Shelhallion, and its effect in deflecting the plumbline from the vertical direction of the gravity of the earth; observed by means of zenith distances of stars on both sides of the mountains, compared to those which should take place under the influence of gravity alone; the observed angular difference was eleven seconds and two-thirds. This, as we have seen, gave 4,715 for the density of the earth, the density of water being unit. But we have seen that proceeding from this scale to the density of the sun and planets, we arrive at densities for them which are considerably at variance with our ideas of solidity, compared to the different kinds of matter around us, and therefore with that state in which we should expect to find the sun or the planets; we therefore find here a subject of inquiry of considerable interest and difficulty for future philosophers; the present state of our knowledge renders it proper to suspend any conclusion, and merely to take the earth as the unit, when compared with the planets, as has been observed in the general remarks upon the solar system.

We may here remark, that all the principles of natural philosophy which we had to apply in our reasonings upon the physical state of the celestial bodies, lead us to the supposition of a gradual increase of density with the approach to the centre of these bodies. The law of this increase is evidently all that we are able to inquire into, in order to solve the problem, and this will have considerable difficulties, on account of the comparatively small depth to which we are able to penetrate into the earth. We have already seen that attempts which have been made with this view, by the observation of the pendulum in mines, have indicated results that appear to lead to a rapid increase, and a much greater mean density, than what has been hitherto adopted, on the faith of the observed deviations of the plumbline.

The theory shows already that the inequalities of the surface of the earth, the depth of the sea, and the different specific gravity of the parts, do not influence the result of the ellip-

ticity of the earth sufficiently to make the difference of moment for astronomical uses; and the theory allows us to consider the earth as yielding, or rather as having originally yielded freely to the combined effect of the gravity and the centrifugal force. Thence, the lineal difference of the radii of the earth will be more observable, because proportionally greater than the variation of the mechanical effect from entire regularity.

A reasoning exactly similar to the above, applies evidently to the analogous question of the actual or specific *Heat* of the earth, taken in mass, or its probably proportional increase towards the centre, upon which our indications are as yet too scanty to authorise us to draw any conclusions.

CHAPTER III.

Of the Atmosphere; Atmospheric Refraction; and the Tides of the Sea and of the Atmosphere.

§. 185. *Matter* presents itself to us under *three* different states, of which each has its distinguishing qualities and mechanical laws. These three states are *Solidity*, *Liquidity*, and *Gaseous Fluidity;* they have each their distinct, and exactly limited, relation to *Gravity*, though matter, as such, is in all these three states equally subject to its influence; and this relation is susceptible of being expressed by mathematical formulæ.

Solidity implies a *Cohesion* between the parts of matter, *superior to Gravity.*

Liquidity presents a state *in exact equilibrium with gravity*, and therefore the most closely following its laws; we might indeed say representing them with exactitude.

Gaseous Fluidity implies *Elasticity;* with the introduction of this additional property of matter we obtain what we may call a *Force* or *Power*, acting in opposition to *Cohesion*, and equally independent of *Gravity*.

To apply these first principles of elementary philosophy to the purpose of our present inquiry, we have only to attend to the simple principles stated in defining the two latter states. *Solidity*, and its effects, we have generally assumed; or rather, we have not had need hitherto to distinguish between the different states of matter; we have had to consider *Mass*, that is, *an aggregate of matter*, acting as *one whole*, produc-

ing a certain mechanical effect, and yielding equally to the general law of gravitation. In this chapter, matter in the two other states comes under our consideration, with the above enumerated distinctive qualities; and the elastic fluid, or atmosphere, surrounding our earth, may be first considered, in order to complete the *general account* of our earth. The peculiar effect of attraction, extraneous to that of the centre of gravity of our earth, which we can observe only upon the *liquid* and the *fluid* parts of our earth, and which we might consider as corresponding to the perturbations in the planetary system, because it acts as an apparent disturbance of the equilibrium of certain component parts of our globe, will furnish a separate point of view under which to treat them.

§ 186. In speaking of the sun, it was necessary to suppose a general knowledge of the existence of our atmosphere, and to state the general idea of the probability, we might say necessity, of it in all celestial bodies. Whether we consider it as that part of earthly matter most remote from the centre of gravity or attraction of the earth, and a constituent part of the same, or as the result of the attraction of any centre of a celestial body upon the *Ether*, or most rare matter, that we may suppose to exist in our solar system, and even the universe at large, condensed by the effect of attraction; we have, from observation and experience, that knowledge of its mechanical laws and effects, which enables us to conduct our investigations in respect to it according to the general principles of natural philosophy.

§ 187. The elastic fluid of our atmosphere is the rarest of the three distinct kinds of matter that surround our earth. The perfectly free motion of its particles among each other, which enables it, by its elasticity, to occupy with equal quantities a larger or smaller space, renders it also compressible by its own weight; that is, by the effect of gravity upon its parts. In consequence of this, its density is continually decreasing with the augmentation of the distance from the surface of the earth, or what is called its altitude. This decrease takes

place in a geometrical ratio, corresponding to the arithmetical ratio of the elevation. From this circumstance has been deduced a very convenient method of determining the elevation of points on the surface of the earth, above the level of the sea, by means of the height of a column of mercury, the heaviest fluid that we have in our ordinary temperatures, being put in equilibrium with the pressure of the atmosphere by means of the *Barometer*. This will at any place, by the length of the column of mercury, indicate the pressure of the whole superincumbent atmosphere, whence any difference observed between two places will correspond to a certain elevation between them; the laws of their mutual dependance, having regard to all the influencing circumstances, have been so well investigated, as to lead to very satisfactory results, when the observations are performed with proper care.

A few remarks will show, that it is an idle pursuit to attempt to determine the actual height of the atmosphere. We have for the law of this research a geometrical series, decreasing from the point at the surface of the sea, where its largest term is variable; the number of terms are unascertainable, and though always approaching to nought, yet never reach it exactly. Of such series we have had examples in all the epochs and movements of the celestial bodies, whose returns are only approximate. In all actions, or results from joint causes, we find a point of equilibrium, gradually approaching; in this case we have the *Elasticity* and the *Gravity* of the air, which act in contrary directions, and under various influences; to assign that limit to them in which they would equilibrate exactly, would be assigning the limit of the atmospere; but gravity we have seen to extend to such distances as would give no absolute limit to its action, and the elasticity of elastic fluids, in the state of rarity in which the atmosphere may be, at the great distance at which we should be willing to place this limit, is yet unknown to us, as much as its real nature. The conclusions which we might draw from the twilight, have reference only

to one of its peculiar properties, that of refracting light; which we must naturally suppose to become observable to us only at a certain density of the atmosphere, which we have more than sufficient reason to believe to be far below its greatest elevation; this would correspond to an elevation of 196844 feet.

The greatest height to which the atmosphere has yet been penetrated, is 23018 feet, being 1640 feet above Chimbonaço, the highest point of the earth in America. This height was obtained by rising in an ærostat; the Barometer was there observed to stand at a little less than 13 inches, while at the sea shore, it is usually about 30 inches. At this elevation, the temperature was 22° below freezing, while at the earth's surface it was 70° above it (Fahrenheit's scale); and the æronauts, Gay Lussac and Arrago, found the state of the atmosphere unfavourable to breathe in.

§ 188. It would carry us too far from the general tenor of this work to enter into the details of the constituent parts, the properties, either chemical or mechanical, of the atmosphere; they are, however, of great interest. As the most extended and most rare part of our globe, the variation of its state is the greatest, and its influences the most extensive; these subjects belong to general physics and to meteorology. Of its general effect, in respect to heat, and as the medium in which we live, there has been said of it as much as may suffice for our object, in speaking of the atmosphere of the sun; the property of it that falls more peculiarly under our consideration, connected with astronomy, is that of refracting the light.

§ 189. Borrowing from the principles of optics the general fact: that the rays of light passing from one transparent medium into another of greater density, under an angle other than the perpendicular to the plane dividing the surfaces, will, at the point of incidence, deviate from their former straight course towards this perpendicular; we must immediately perceive that the density of our atmosphere gradually increasing

with the proximity to the earth, a constant deviation of the light from the outside until it reach our eyes must take place. Any celestial phenomenon will therefore appear in a place different from that in which it really is. The ray of light from any celestial body will describe, under this influence, a curve in our atmosphere, which, ultimately meeting our eyes, will present itself to us as coming in the direction of a tangent to this curve at our eye. This deviation will evidently take place in a plane perpendicular to the horizon, and therefore affect every vertical angle observed. The curve bending more and more towards the earth, the tangent to it under which we observe the ray of light will, therefore, be more elevated than the direct line. The angle between this tangent and the direct line to the celestial body is called *Astronomical Refraction*. If we knew the altitude of the atmosphere, and its exact nature in regard to this deviation of light, with the law of its progress according to the increasing density, we might calculate the refraction theoretically, as the change occasioned by the angle of incidence is founded upon simple geometrical principles; but the remarks just made upon these subjects show, that here we must appeal entirely to observation for the elementary number that is employed in all the calculations, which in astronomy is called the *Constant Factor;* this has been of late determined with considerable accuracy and care, as have also the theoretical laws, with the co-operating influences. As the variation depending on the angle under which the ray of light passes the atmosphere, is so much the greater the more this ray of light differs from the perpendicular, the refraction at the horizon is the greatest; the quantity or angle of this *horizontal refraction* is given in TABLE II. for several planets. At the Zenith it becomes nothing; because the rays of light falling upon the different atmospheric layers, of different successive densities, in a perpendicular direction, no deviation or refraction is occasioned.

§ 190. Knowing now that the principle which we have used in our observations, to determine the positions of

celestial bodies in relation to the zenith, is affected by *Refraction*, we are obliged to obtain its determination by a method, which, as we might say, neutralizes this effect by its mode of operation. For this purpose, are principally used observations of such fixed stars, as, on account of their distance from the pole being less than the latitude of the place of observation, will pass the meridian, on both sides of the pole above the horizon. It is easily seen that, as the fixed stars do not change their positions to any perceptible amount between two such passages, stars so situated give, by the difference of the results of their altitudes, when referred to the invariable polar distance, the difference of the effect of the astronomical refraction in their two different passages. But the different influence of the different state of the atmosphere, which complicates the appearances, must be taken into account to determine the laws of the change, as well in proportion to the altitudes, as to the causes producing changes in the atmosphere; these effects are determined in accordance with mathematical principles, for the different states of the atmosphere, as indicated by the *Barometer* and the *Thermometer*, the two most perfectioned meteorological instruments that we have at present. The variability of the atmosphere being independent of the observation, or the position of the celestial body, this influence constitutes a peculiar correction to be applied to the effect of the refraction according to the given case.

§ 191. In investigating the principles of the perturbations, we have seen that the effect of an attraction extraneous to that which the celestial body obeys in its revolution, or if we may say so, of a minor or partial gravitation towards others, occasions an enlargement of the ellipse of revolution, when the attracting point is outside of the same. We have seen the most marked effect of this extraneous attraction in the orbit which the moon describes around the earth, as affected by the attraction of the sun. We have also seen that notwithstanding the small mass of the moon, its effect upon the earth was remarkably great in its consequences.

Having stated at the beginning of this chapter, the fact of three different states of matter on our globe, and the variety they present in their relation to gravity, we might *a priori* be induced by the confrontation of these two facts, to inquire into the possible difference of the influence of an attraction other than that of the centre of the earth upon each of these states. In the solid state, the *Cohesion* of the parts, joined to their density, as it presents to us difficulties in their separation, and requires more power to overcome their gravity, we have also to expect it to be more free from any change of shape from the lesser extraneous attractions; indeed we also find in nature no ascertainable effect, and the smallness of the effect observed upon the liquid part of our globe, the seas, might be considered as indicating that the effect upon the solid part of the earth must be entirely unobservable.

But, as Liquidity presents a state in perfect equilibrium with gravity, the great mass of water upon our earth, the sea, in a change of its form, and of the relative position of its parts, must exhibit the effect of the attraction of *the moon* and *the sun*, corresponding to the proportion their attraction bears to the earth itself. In this way the form of the surface of the sea is modified into that of an *elongated ellipsoid*, presenting the form resulting from an equilibrium between the co-operating attractions. The moon and the sun having each necessarily its own separate effect, the intermixture of the two figures produces an intermediate form, which varies daily, in consequence of the different angle of the directions under which the sun and moon act upon the earth.

If we suppose the sea to surround the whole earth, it would, within the time of its revolution in relation to the sun and the moon, present this form, successively changing its position in proportion with the relation of their diurnal revolution. The elevation of the water towards each of the attracting points, will cause a depression on both sides, at right angles to that line, and another elevation on the side of the earth opposite to the attracting point, or as we have seen above, to the

direction resulting from the combined attraction of the sun and moon.

Thus the surface of the sea always presents four distinct points, of which two, diametrically opposite to each other, are elevated above the natural level; and two others, likewise diametrically opposite, and at right angles to the two preceding ones, are depressed below this mean level. Every point of the circumference of the earth on the sea shore, therefore, sees, twice in the course of one rotation, or day, the waters of the sea swelling against the land, and again twice receding from it; the first is called the *Flood*, the second the *Ebb, of the sea*, and the whole phenomenon constitutes what is called the *Tides*.

§ 192. In the tides, then, we again have a combined effect of the attraction of the sun and the moon, subject to all the variations of the relative magnitude of their influence, which varies naturally according to their distance, and the angle under which they act. On account of the proximity of the moon, compared with that of the sun, the effect of the moon upon the tides is about three times as great as that of the sun, the tides therefore, generally speaking, follow the course of the moon; the action of the sun modifies this effect, but this modification, though of great importance, is discernible rather in changing thec onsequences of this effect, than in the general course of the phenomenon.

We are aware, from what has been said heretofore, that the variation of the distance of the moon, or even of the sun, the different angles the direction of their action makes with the equator of the earth, considered as the plane of rotation, in which the principal action of the tides must take placé, and, therefore, the variation of the angle the moon's direction makes with the plane of the ecliptic, in which action of the sun takes place, which are all variable quantities, must influence the quantity of the tides. Hence arise a variety of epochs similar to what we have seen in other parts of astronomy; we should therefore again have a variety of details to consi-

der, and several epochs to determine, which cross each other, if we would enter into the details which it furnishes to analytic calculation; of these, however, a few of the principal results may suffice to give a clear idea of this subject, which is all that is to be done in this place.

§ 193. The general phenomenon that is first apparent, shows us the waters of the sea rising for somewhat more than six hours, and falling an equal length of time, to rise and fall again with little variation during the other half of the day, so as to coincide again the next day with the same relative situation of the moon; these changes occupy in two such revolutions, about fifty minutes and a half more than one day.—If we commence daily observations upon this phenomenon at the time of the conjunction of the sun and moon, that is, at *New moon*, we shall soon observe that the magnitude of its effect is the greatest at that time, and diminishes till the sun and the moon appear at right angles to each other, or in quadrature. The reason of this is as follows: the two lines of their attraction are then perpendicular to each other, and therefore the resulting elongation towards the moon is only the difference of the two effects united at first. From this time, the magnitude of the tides will increase again until the *Full moon*, when the action of the sun and moon being in the same line, but in opposite directions, will both again tend to lengthen the same axis of the elongated ellipsoid, and produce an effect nearly equal to that of the new moon.

These variations of the magnitudes of the tides will again vary in consequence of the combination of the changeable distances of the sun and moon from the earth, and be so much the greater, the smaller they are. That in this also the angle with the equator, under which they act, comes into consideration, is self-evident.

Observation has also shown a decided increase of the tides at the times of the equinoxes and solstices, which added to the before mentioned circumstances, produce the greatest elevations of the tides at the new moon and full moon, nearest to these epochs.

§ 193. The magnitude of the tides, though it preserves the above laws of variation, is however different at every different places, in its absolute quantity, and varies, between high and low water, from three to sixty five feet. It generally appears to be higher in higher latitudes, where the line of attraction is under a less angle with the surface of the sea, and thence acts under more favourable circumstances against the attraction of the earth upon it; the maximum falling in 79½ degrees of latitude. But this quantity is generally so much under the influence of local circumstances, such as the depth, width, and configuration of the channel it may have to pass, and the configuration of the shore upon which it takes place, the currents of the sea, and the prevailing winds, &c., that no general law is as yet discoverable.

Theory alone would give it directly in proportion to the magnitude of the attracting mass, and inversely as the square of the distance, combined with the attracting power of the earth, under various angles. In order then to obtain a series of comparable observations, with a view to the investigation of the theory, it is necessary to continue them at the same place, under the same circumstances as to local influence during the whole time of one or more *Revolutions of the moon's orbit*; that all the situations of the moon, in relation to the sun, may be compared in every respect. There will still, after this, remain the influence of the winds as an accidental circumstance to be noticed, of which no account can be kept, but which is considerable.

Though the action of gravity is propagated with more rapidity than that of light, the greatest effect of the tides follows the moment when the position of the moon would indicate it at various intervals. In this, as in the magnitude of the tides, the influence of local circumstances is very great, therefore this time of retardation is usually indicated for the different seaports, as it takes place at the time of full and new moons; this must of course be also subject to the variation of the force producing the tides, and therefore varies with it.

§ 195. The phenomenon of the tides presents such a direct connexion with the attraction of the moon, that it can be again employed inversely in the determination of some of the other results of this attraction, as for instance, the proportion between the lunar and the solar nutation, and other similar data; for instance, the mass of the moon has been determined by it to be $\frac{1}{68,7}$, or 0,014556 of that of the earth.

§ 196. Seeing that such an effect is produced upon the sea by the extraneous attraction of the sun and moon, we are convinced that a similar one of much greater lineal extent must take place in the atmosphere, the physical state and properties of which render its influence so much more easy; and on account of its full freedom all round the earth, its motion must be constant and unimpeded, the differences of elevation of the surface of the earth being too minute to have any remarkable influence in this respect. The atmosphere has been considered to form an ellipsoid around the earth, the proportion between the polar and equatorial diameter of which is as 2 to 3. The effect of these tides must therefore be different at different latitudes.

The influence of the varied temperature, whether local or general, shows us a great variation in the barometer, the instrument by which we measure the pressure on what we call the temporary height of the atmosphere; however, the variability of its other mechanical powers, and its chemical state, no doubt, have a great share in producing the variations which we observe in the barometer. The number of circumstances and influences combining, necessarily require an extension of the series of observations leading to a theory; the circumstances must also be determined for a great length of time, and a great variety of places, they require the co-operation of many persons, and a close and frequent daily attendance. This has not, however, impeded this interesting inquiry; registers continued for long series of years have been made, and the corresponding observations are multiplying constantly; but an accurate systematic result is as yet so much more difficult,

as the chemical influences in the atmosphere appear to be as great, if not greater than the mechanical ones.

The fact of the daily tide in the atmosphere has been already discovered to arise both from the sun and the moon; but observations must be made upon this subject, which essentially distinguish the phenomenon of the tides of the atmosphere from those of the sea. We cannot observe it by an actual lineal measure, like the tides of the sea. An extraneous attraction upon an elastic fluid will not draw it from its place in the same state of density in which it is there, as is the case in water; but it will, at the same time with this extension, diminish its density, and we might even expect this to be the greater part of the effect. Hence the equilibrium which we observe in the column of mercury of the barometer, can only indicate a fractional effect of the atmospheric tide, and this is shown by a very reduced lineal magnitude; namely, in the proportion of the height of the atmosphere to the column of mercury. All this is likewise subject to the modification of what we may call this height of the atmosphere, which, as seen above, we are deprived of direct means to ascertain. We may hence conclude, that the tides of the atmosphere, which principles indicate as unavoidably existing, consist rather in a change of its density, or at least more in this than in an extension of the denser part of it to a greater elevation above the surface of the earth; this fact, therefore, renders the investigation a much more delicate subject. In this, as well as in all other subjects in Meteorology, we are not yet prepared to obtain accurate results; and a deeper inquiry into them lies out of the limits of our aim in this volume.

CORRECTIONS.

Page 3, line 30, for "apparent," read "apparently different."

13, 27, for "1800," read "1801."

—, 28, for "1813," read "1820."

16, 21, for "given," read "represented."

18, 22, after "exterior," add "to each other."

20, 3, for "1 and 2," read "2 and 3."

—, 4, for "Fig. 1," read "Fig. 2."

—, 8, for "Fig. 2," read "Fig. 3."

—, 23, for "Fig. 2," read "Fig. 3."

34, 10, for "apparent angular," read "apparent greatest angular."

35, 36, for "and approaches," read "and he approaches."

36, 36, for "in," read "in her."

38, 19, for "increases," read "decreases."

—, 29, for "most," read "more."

—, 31, for "from her first appearance," read "since her last appearance."

42, 14, for "Venus." read "Venus; and."

43, 28, for "superior," read "superior to each other."

46, 22, for "even," read "if it."

53, 30, for "ascertained," read "resulting."

56, 7, for "relation," read "rotation."

—, 15, for "eighth," read "eight."

63, 7, for "supposition that," read "supposition of that."

—, 10, for "$\frac{22}{100000}$," read "$\frac{12}{100000}$."

75, 3, omit "$\frac{1}{292}$."

85, 36, for "and," read "acting."

94, 19, for "place," read "direction."

110, 15, for "of the," read "by a."

—, 33, for "t," read "t'."

111, 3, for "t," read "t'."

119, 7, for "the," read "their."

122, 5, for "Orbit," read "plane of the Equator."

—, 22, for "direct," read "full."

124, 24, for "as equated," read "of Jupiter."

—, 32, after "expect," add "it."

125, 28, after "times," add "that."

137, 22, for "of that of," read "of that of the orbit of."

138, 10, for "Table II." read "Table III."

—, 13, for "distance," read "distance from us."

141, 1, for "calculated," read "contained."

157, 2, for "with their orbits," read "with the plane of their orbits."

162, 33, for "stars," read "stars;"

—, 35. for "time;" read "time,"

164, 24, for "sphere;" read "sphere,"

168, 24, after "such," omit the comma.

182, 25, for "have," read "leave."

183, 12, for "nebulæ," read "nebula."

—, 19, do do.

187, 2, after "finds," add "them subject to."

199, 19, for "point;" read "point,"

200, 5, for "and be," read "and there be."

205, 5, after "indicate," add "with satisfactory accuracy."

207, 4, for "elliptic," read "ellipsoidic."

—, 28, after "present," add "us."

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Table V. presents more properly what is called in Astronomy a *Table of Double Entry*, or similar to the Pithagorean Multiplication Table; the planet, as ranged in the left hand column, will see each of those marked at the heads of the columns as indicated beneath this second planet, when in their greatest elongation from the Sun.

Wherever double numbers appear in the tables for the same result, the upper indicates the best determinations followed until very lately, and the lower denotes the newest result obtained; those numbers only that are different are written, the —— indicating the repetition of the upper numbers.